EMS Tracts in Mathematics 5

EMS Tracts in Mathematics

This series includes advanced texts and monographs covering all fields in pure and applied mathematics. *Tracts* will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.

1 Panagiota Daskalopoulos and Carlos E. Kenig, *Degenerate Diffusions*
2 Karl H. Hofmann and Sidney A. Morris, *The Lie Theory of Connected Pro-Lie Groups*
3 Ralf Meyer, *Local and Analytic Cyclic Homology*
4 Gohar Harutyunyan and B.-Wolfgang Schulze, *Elliptic Mixed, Transmission and Singular Crack Problems*

Gennadiy Feldman

Functional Equations and Characterization Problems on Locally Compact Abelian Groups

European Mathematical Society

Author:

Gennadiy Feldman
Mathematical Division
B. Verkin Institute for Low Temperature Physics and
Engineering of the National Academy of Sciences of Ukraine
47 Lenin Ave.
Kharkov, 61103, Ukraine

E-mail: feldman@ilt.kharkov.ua

2000 Mathematics Subject Classification: 60-02, 60B15, 62E10; 43A05, 43A25, 43A35, 39B52

Key words: Locally compact Abelian group, Gaussian distribution, characterization problems of mathematical statistics, functional equation

ISBN 978-3-03719-045-6

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the authors' TEX files: I. Zimmermann, Freiburg
Printed in Germany

9 8 7 6 5 4 3 2 1

Preface

Characterization problems in mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions in these variables. One of the famous examples of a characterization problem is the classical Kac–Bernstein theorem ([65], [13]). This theorem characterizes a Gaussian distribution by the independence of the sum $\xi_1 + \xi_2$ and of the difference $\xi_1 - \xi_2$ of independent random variables ξ_j. Taking into account that the characteristic function of the random variable ξ_j with distribution μ_j is the expectation $f_j(y) = \hat{\mu}_j(y) = \mathbf{E}[e^{i\xi_j y}]$, it is easily verified that the Kac–Bernstein theorem is equivalent to the statement that, in the class of normalized continuous positive definite functions, all solutions to the Kac–Bernstein functional equation

$$f_1(u+v)f_2(u-v) = f_1(u)f_1(v)f_2(u)f_2(-v), \quad u, v \in \mathbb{R},$$

are of the form $f_j(y) = \exp\{-\sigma y^2 + ib_j y\}$, where $\sigma \geq 0$, and $b_j \in \mathbb{R}$.

The Kac–Bernstein theorem was the first among characterization theorems where independent linear forms of independent random variables ξ_j under different restrictions on ξ_j were studied. These studies were completed with the following Skitovich–Darmois theorem ([98], [23]): Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, and α_j, β_j be nonzero real numbers. If the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent, then all random variables ξ_j are Gaussian. Just as in the case of the Kac–Bernstein theorem, the Skitovich–Darmois theorem is equivalent to the statement that, in the class of normalized continuous positive definite functions, all solutions to the Skitovich–Darmois functional equation

$$\prod_{j=1}^{n} f_j(\alpha_j u + \beta_j v) = \prod_{j=1}^{n} f_j(\alpha_j u) \prod_{j=1}^{n} f_j(\beta_j v), \quad u, v \in \mathbb{R},$$

are of the form

$$f_j(y) = \exp\{-\sigma_j y^2 + ib_j y\}, \tag{1}$$

where $\sigma_j \geq 0$ and $b_j \in \mathbb{R}$.

The Skitovich–Darmois theorem was generalized by Ghurye and Olkin ([54]) to the multivariable case when, instead of random variables, random vectors ξ_j in the space $\mathbb{R}^m$ are considered, and coefficients of the linear forms L_1 and L_2 are nonsingular matrices. In this case the independence of L_1 and L_2 also implies that all independent random vectors ξ_j are Gaussian. The proof of the Ghurye–Olkin theorem is also reduced to solving the corresponding functional equation. It should be noted that nonsingular matrices are topological automorphisms of the group $\mathbb{R}^m$.

Next Heyde proved the following result ([61]): Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, and let α_j, β_j be nonzero real numbers such that

$\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all ξ_j are Gaussian. This statement is closely related to the Skitovich–Darmois theorem. The Heyde theorem is also equivalent to the assertion that, in the class of normalized continuous positive definite functions, all solutions to the Heyde functional equation

$$\prod_{j=1}^{n} f_j(\alpha_j u + \beta_j v) = \prod_{j=1}^{n} f_j(\alpha_j u - \beta_j v), \quad u, v \in \mathbb{R},$$

are of the form (1).

In the last 30 years much attention has been devoted to generalizing of the classical characterization theorems into various algebraic structures such as locally compact Abelian groups, Lie groups, quantum groups, symmetric spaces (see e.g., [2], [29], [31]–[44], [46], [47], [49], [52], [55], [56], [64], [78]–[82], [84]–[87], [91], [93], [94]).

These investigations were motivated, first of all, by the desire to find the natural limits for possible extensions of the classical results. The present book is devoted to generalization of the Kac–Bernstein, Skitovich–Darmois, and Heyde characterization theorems to the case where independent random variables take values in a second countable locally compact Abelian group X, and coefficients of linear forms are topological automorphisms of X. It turns out that the possibility to prove a characterization theorem for X not only depends on the structure of X but also determines its structure. For example, assume that independent random variables ξ_1 and ξ_2 take values in X and their characteristic functions do not vanish, then the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that ξ_j are Gaussian if and only if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Note that in the case of groups as well as in the classical case, the proof of this theorem can be reduced to solving of some functional equation in the class of normalized continuous positive definite functions on the character group of X.

We describe and comment on the main contents of the book.

Chapter I contains mainly well-known facts from abstract harmonic analysis and the theory of infinite Abelian groups. In Section 1 we give the basic definitions and consider some examples of locally compact Abelian groups. In particular, we describe all subgroups of the additive group of the rational numbers $\mathbb{Q}$, the groups of p-adic integers Δ_p, and $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$. We formulate structure theorems for various classes of locally compact Abelian groups and describe the topological automorphism groups of the groups $\mathbb{R}^n$, $\mathbb{T}^n$, Δ_p, $\Sigma_{\boldsymbol{a}}$. The basic results of Ulm theory for countable p-primary Abelian groups are also presented. In Section 2 we discuss some aspects of probability distributions on locally compact Abelian groups (the Bochner theorem, properties of characteristic functions, the Lévy–Khinchin formula, idempotent distributions).

Chapter II is devoted to Gaussian distributions on a locally compact Abelian group X. In Section 3 we define Gaussian distributions and study their properties. In Section 4 we describe locally compact Abelian groups where every Gaussian distribution has only Gaussian factors (the group analogue of the Cramér theorem of decomposition of a Gaussian distribution). In Section 5 we study properties of continuous polynomials on locally compact Abelian groups. We describe locally compact Abelian groups where any distribution μ with a characteristic function of the form $\hat{\mu}(y) = e^{P(y)}$, where $P(y)$ is a continuous polynomial, is Gaussian (the group analogue of the Marcinkiewicz theorem). The group analogues of the Cramér and Marcinkiewicz theorems are basic tools for proving characterization theorems in Chapters III–VI. In Section 6 we consider Gaussian distributions in the sense of Urbanik, i.e., such distributions on X which any character transforms into Gaussian distributions on the circle group $\mathbb{T}$. We describe locally compact Abelian groups for which the class of Gaussian distributions coincides with the class of Gaussian distributions in the sense of Urbanik.

In Chapter III we study distributions of independent random variables ξ_1 and ξ_2 taking values in a locally compact Abelian group X such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. In Section 7 we describe all groups X where such distributions are invariant with respect to a compact subgroup K of X and that, under the natural homomorphism $X \mapsto X/K$, induce Gaussian distributions on the factor group X/K. This is the widest subclass of locally compact Abelian groups on which the Kac–Bernstein type theorem can be proved. It consists of all groups X having the connected component of zero without elements of order 2.

If the connected component of zero of a group X contains elements of order 2, then for such groups the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. We present a solution to this problem in Section 8 for the group $\mathbb{R} \times \mathbb{T}$ and the $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$. In Section 9 we study distributions of independent identically distributed random variables ξ_1 and ξ_2 taking values in a locally compact Abelian group X such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent (Gaussian distributions in the sense of Bernstein).

Chapters IV and V are devoted to some group analogues of the Skitovich–Darmois theorem. Let ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables taking values in a locally compact Abelian group X, and α_j, β_j be topological automorphisms of X. Put $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$.

In Chapter IV we assume that the characteristic functions of independent random variables ξ_j do not vanish. We prove in Section 10 that independence of the linear forms L_1 and L_2 implies that all ξ_j are Gaussian if and only if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Under the condition that the characteristic functions of ξ_j do not vanish, this is the widest subclass of locally compact Abelian groups on which the Skitovich–Darmois theorem can be extended.

Assume that a group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$. Then the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and having the property that the linear forms L_1 and L_2 are independent. The remainder of Chapter IV is devoted to solution of this problem. It turns out that if the characteristic functions of independent random variables ξ_j with distributions μ_j do not vanish and L_1 and L_2 are independent, then the distributions μ_j can be replaced by their shifts μ'_j in such a way that all μ'_j are supported in the connected component of zero of the group X. Hence the problem can be reduced to the case when X is connected. An important characteristic of a connected locally compact Abelian group X is its dimension $\dim X$. If $\dim X = 1$, then X is topologically isomorphic to one of the following groups: the real line $\mathbb{R}$, an $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ or the circle group $\mathbb{T}$. If $\dim X = 2$, then either X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$ or X is topologically isomorphic to one of the following groups: $\mathbb{T}^2$, $\mathbb{R}\times\mathbb{T}$ or $\Sigma_{\boldsymbol{a}}\times\mathbb{T}$. Assume that the number of random variables ξ_j is equal to 2. In Section 11 we describe for the two-dimensional torus $\mathbb{T}^2$ all possible distributions of the random variables ξ_j in the case when linear forms L_1 and L_2 are independent. Generally speaking, these distributions are not Gaussian, and we describe, in particular, all topological automorphisms α_j, β_j of the two-dimensional torus $\mathbb{T}^2$ for which the corresponding distributions are Gaussian. In Section 12 we solve the same problem for the groups $\mathbb{R}\times\mathbb{T}$ and $\Sigma_{\boldsymbol{a}}\times\mathbb{T}$.

In Chapter V we omit the assumption that the characteristic functions of independent random variables ξ_j do not vanish and suppose that ξ_j take values in different classes of locally compact Abelian groups (finite, discrete, discrete torsion, compact totally disconnected, etc.) Since Gaussian distributions on a totally disconnected group are degenerate, idempotent distributions play an important role on such groups. It turns out that, in contrast to the classical situation, there are essential distinctions between the cases when we deal with linear forms of two random variables, of three random variables, or of $n\geq 4$ random variables. In the group situation some new effects appear which do not hold on the real line. To show this consider the following example. Let $\mathbb{Z}(5)$ be the group of residues modulo 5, and ξ_j, $j=1,2,\dots,n$, $n\geq 2$, be independent random variables with values in $\mathbb{Z}(5)$. If $n=2$ and the linear forms L_1 and L_2 are independent, then all random variables ξ_j have idempotent distributions. If $n=3$ and the linear forms L_1 and L_2 are independent, then we can only assert that at least one of the random variables ξ_j has an idempotent distribution. On the other hand for every $n\geq 4$ there exist independent random variables ξ_j, $j=1,2,\dots,n$, taking values in $\mathbb{Z}(5)$ and automorphisms α_j, β_j of $\mathbb{Z}(5)$ such that the linear forms L_1 and L_2 are independent, but all ξ_j have non-idempotent distributions.

In Section 13 we consider two independent random variables ξ_1 and ξ_2. We prove that if X is a discrete group and the linear forms L_1 and L_2 are independent, then ξ_1 and ξ_2 have idempotent distributions. We also describe compact totally disconnected groups for which this property holds true. We prove that the Skitovich–Darmois theorem fails for compact connected groups. In Section 14 we study the case when the number n of random variables ξ_j, $j=1,2,\dots,n$, is greater than 2. First we assume that the ξ_j take values in a finite group. Then we study the case when X is either a compact totally disconnected group or a discrete torsion group. We describe in both cases the groups X for which independence of the linear forms L_1 and L_2 implies that either all random variables ξ_j have idempotent distributions or at least two of the random variables ξ_j have idempotent distributions, or at least one of the random variables ξ_j has an idempotent distribution. Further we consider an arbitrary number $n\geq 4$ of random variables ξ_j, and X is assumed to be either a discrete torsion group or a compact group. We note that our proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n = 2$, in contrast to the proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n > 2$, does not use the Ulm theory.

In Section 15 we consider independent random variables ξ_1 and ξ_2 taking values in an **a**-adic solenoid $\Sigma_{\boldsymbol{a}}$. We describe all possible distributions of random variables ξ_j assuming that the linear forms L_1 and L_2 are independent. The result depends on both an **a**-adic solenoid and topological automorphisms α_j, β_j.

In Chapter VI we study group analogues of the Heyde theorem. Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables taking values in a locally compact Abelian group X. Let α_j, β_j be topological automorphisms of X satisfying the condition

(i) $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1}$ are topological automorphisms of X for all $i \neq j$.

Let $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$. Assume first that the characteristic functions of the independent random variables ξ_j do not vanish. In Section 16 we prove that symmetry of the conditional distribution of the linear form L_2 given L_1 implies that all ξ_j are Gaussian if and only if X contains no elements of order 2. This result can not be improved. If a group X contains elements of order 2, then the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and having the property that the conditional distribution of the linear form L_2 given L_1 is symmetric. We assume that on the group X there exist topological automorphisms α_j, β_j satisfying condition (i). A simple example of such a group is the two-dimensional torus $X = \mathbb{T}^2$, and even in this case the problem is very interesting. It turns out that the distributions which are characterized by the symmetry of the conditional distribution of the linear form L_2 given L_1 are convolutions of Gaussian distributions concentrated on a dense one-parameter subgroup of $\mathbb{T}^2$ and distributions supported in the subgroup of $\mathbb{T}^2$ generated by elements of order 2.

In Section 17 we drop the assumption that the characteristic functions of independent random variables ξ_j do not vanish. We first study the case when ξ_j take values in a finite group and then ξ_j take values in a discrete group X. In the case when X is discrete and the number of random variables ξ_j is 2, we prove in particular that the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that the random variables ξ_1 and ξ_2 have idempotent distributions if and only if the group X contains no elements of order 2.

In the appendix we study the Kac–Bernstein and Skitovich–Darmois functional equations on locally compact Abelian groups in the classes of continuous and measurable functions.

The author would like to thank the B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine for providing excellent facilities for writing this book.

March 2008 G. M. Feldman

Contents

Chapter I
Preliminaries

1 Locally compact Abelian groups

We present in this section a summary of the results of abstract harmonic analysis and theory of infinite Abelian groups which we use in the book. The monographs by Hewitt and Ross ([59]) and Fuchs ([50], [51]) are our main sources. Unless the contrary is stated, when we speak of groups, we consider only locally compact Abelian groups.

1.1 Some definitions and notation. Let X be a locally compact Abelian group. Unless otherwise stated, we use the additive notation for the operation in X and denote by "0" the neutral element of X. A topological isomorphism of groups, i.e., an algebraic isomorphism which is a homeomorphism we denote by the symbol "$\cong$". Let x be an element of X. The element x is said to be *compact*, if the smallest closed subgroup of X containing x is compact. Denote by b_X the set of all compact elements of X, and denote by c_X the connected component of zero of X. If every element of X has finite order, then X is called a *torsion group*. If every element of X except zero has infinite order, then X is called a *torsion-free group*. Let $f : \mathbb{R} \mapsto X$ be a continuous homomorphism. The image $f(\mathbb{R})$ is called a *one-parameter subgroup* of X.

A finite subset $\{x_1, \dots, x_n\}$ of a group X is said to be *independent* if it does not contain zero, and if for any integers $k_1, \dots, k_n$ the equality $k_1x_1 + \cdots + k_nx_n = 0$ implies that $k_1x_1 = \cdots = k_nx_n = 0$. An infinite subset A of X is said to be *independent* if every finite subset of A is independent. The cardinality of a maximal independent subset of the group X containing only elements of infinite or prime power order is called the *rank* of X. Denote by $r(X)$ the rank of X and by $\dim X$ the dimension of a connected group X.

Let $\{X_\iota : \iota \in \mathcal{I}\}$ be a nonvoid family of groups. Denote by $\mathbf{P}_{\iota\in\mathcal{I}} X_\iota$ the *direct product of the groups* X_ι, i.e., the group coinciding with the Cartesian product of the sets X_ι, and with the coordinate-wise operation. Let $\mathbf{P}^*_{\iota\in\mathcal{I}} X_\iota$ be the subset of all $(x_\iota) \in \mathbf{P}_{\iota\in\mathcal{I}} X_\iota$ such that $x_\iota = 0$ for all but a finite set of indices ι. Then $\mathbf{P}^*_{\iota\in\mathcal{I}} X_\iota$ is a subgroup of $\mathbf{P}_{\iota\in\mathcal{I}} X_\iota$. It is called the *weak direct product of the groups* X_ι. If $X_\iota = X$ for all $\iota \in \mathcal{I}$, then the direct product of the groups X_ι we denote by $X^{\mathfrak{n}}$, and the weak direct product of the groups X_ι we denote by $X^{\mathfrak{n}*}$, where $\mathfrak{n}$ is the cardinal number of the set $\mathcal{I}$. Let $\{K_\iota : \iota \in \mathcal{I}\}$ be a nonvoid family of compact groups. The group $\mathbf{P}_{\iota\in\mathcal{I}} K_\iota$ we always consider in the product topology. By the Tychonoff theorem the group $\mathbf{P}_{\iota\in\mathcal{I}} K_\iota$ is also compact. If $\{D_\iota : \iota \in \mathcal{I}\}$ is a nonvoid family of discrete groups, then the group $\mathbf{P}^*_{\iota\in\mathcal{I}} D_\iota$ we always consider in the discrete topology.

Let n be an integer. Denote by f_n the mapping of X into X defined by $f_nx = nx$. Put $X_{(n)} = \operatorname{Ker} f_n$ and $X^{(n)} = f_n(X)$. A group X is said to be *a Corwin group* if $X^{(2)} = X$. A group X is said to be *divisible*, if $X^{(n)} = X$ for all natural n.

If G is a subgroup of X, then we denote either by $x + G$ or by $[x]$ elements of the factor group X/G. Let A be a subset of X. The set of all elements of the form $x = l_1x_1 + \cdots + l_kx_k$, where $x_j \in A$ and l_j are integers, is said to be the *subgroup of X generated by A*. Let A and B be subsets of X. Denote by $A + B$ the set $A + B = \{x \in X : x = a + b, a \in A, b \in B\}$. If A is a subset of X, then denote by $\overline{A}$ the closure of A. For a finite set A denote by $|A|$ the number of elements of A.

Unless otherwise stated, the word "function" means a mapping taking either real or complex values. Let Y be an arbitrary Abelian group, $f(y)$ be a function on Y, h be an arbitrary element of Y. Denote by Δ_h the *finite difference operator*

$$\Delta_h f(y) = f(y + h) - f(y).$$

A function $f(y)$ on Y is called a *polynomial* if

$$\Delta_h^{n+1} f(y) = 0$$

for some n and for all $y, h \in Y$. The minimal n for which this equality holds is called the *degree* of the polynomial $f(y)$. A function $f(y)$ on the group Y is said to be *normalized* if $f(0) = 1$.

We use the symbols $\boldsymbol{\omega}$ and $\aleph_0$ to denote the least infinite ordinal number and the least infinite cardinal number respectively. Denote by $\mathbb{N}$ the set of natural numbers, and denote by $\mathscr{P}$ the set of prime numbers. The set of complex numbers we denote by $\mathbb{C}$.

1.2 Examples of locally compact Abelian groups. Let us consider the most important locally compact Abelian groups which we need for the study of characterization problems. Note that we consider all finite groups in the discrete topology.

(a) $\mathbb{R}$: the additive group of real numbers with the natural topology of the real line.

(b) $\mathbb{T}$: the circle group (the one-dimensional torus), i.e., $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with multiplication as an operation and with the usual topology.

(c) $\mathbb{Z}$: the additive group of integers (the infinite cyclic group) with the discrete topology.

(d) $\mathbb{Z}(m)$: the group of residue modulo m (the finite cyclic group). The elements of the group $\mathbb{Z}(m)$ we denote by $\{0, 1, \dots, m-1\}$. We note that the group $\mathbb{Z}(m)$ is isomorphic to the multiplicative group of mth roots of unity. We retain for this group the notation $\mathbb{Z}(m)$, and its elements will be denoted by $\exp\{2\pi ik/m\}$, $k = 0, \dots, m-1$.

(e) Let p be a prime number. Consider a set of rational numbers of the form $\{k/p^n : k = 0, \dots, p^n - 1, n = 0, 1, \dots\}$ and denote this set by $\mathbb{Z}(p^\infty)$. If we define the operation in $\mathbb{Z}(p^\infty)$ as addition modulo 1, then $\mathbb{Z}(p^\infty)$ is transformed into an Abelian group which we consider in the discrete topology. Obviously, this group is isomorphic to the multiplicative group of p^nth roots of unity, where n goes through the nonnegative integers, considered in the discrete topology. This group we also denote by $\mathbb{Z}(p^\infty)$, and its elements by $\exp\{2\pi ik/p^n\}$, $k = 0, \dots, p^n - 1, n = 0, 1, \dots$.

(f) Let $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$ be a fixed but arbitrary infinite sequence of integers, where each of a_n is greater than 1. We define the additive group of $\boldsymbol{a}$-*adic* *integers* $\Delta_{\boldsymbol{a}}$. As a set $\Delta_{\boldsymbol{a}}$ coincides with the Cartesian product $\mathbf{P}_{n=0}^{\infty}\{0, 1, \dots, a_n - 1\}$. Consider $\boldsymbol{x} = (x_0, x_1, x_2, \dots)$, $\boldsymbol{y} = (y_0, y_1, y_2, \dots) \in \Delta_{\boldsymbol{a}}$, and define the sum as follows. Let $x_0 + y_0 = t_0 a_0 + z_0$, where $z_0 \in \{0, 1, \dots, a_0 - 1\}$, $t_0 \in \{0, 1\}$. Assume that the numbers $z_0, z_1, \dots, z_k; t_0, t_1, \dots, t_k$ have been already determined. Let us set then $x_{k+1} + y_{k+1} + t_k = t_{k+1} a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, \dots, a_{k+1} - 1\}$, $t_{k+1} \in \{0, 1\}$. This defines by induction a sequence $\boldsymbol{z} = (z_0, z_1, z_2, \dots)$. The set $\Delta_{\boldsymbol{a}}$ with the addition defined above is an Abelian group, whose neutral element is the sequence in $\Delta_{\boldsymbol{a}}$ that is identically zero. Consider $\Delta_{\boldsymbol{a}}$ in the product topology. The obtained group is called the $\boldsymbol{a}$-adic integers. The group $\Delta_{\boldsymbol{a}}$ is compact and totally disconnected ([59], (10.2)).

If all of the integers a_n are equal to some fixed prime integer p, we write Δ_p instead of $\Delta_{\boldsymbol{a}}$, and call this object the group of *p-adic integers*. Each element $\boldsymbol{x} = (x_0, x_1, x_2, \dots) \in \Delta_p$ is thought of as a formal power series $\sum_{n=0}^{\infty} x_n p^n$. The addition of formal power series defined in the usual way corresponds to the addition of the corresponding sequences. One can define a multiplication in Δ_p in the natural way as the multiplication of formal power series. Then Δ_p is transformed into a commutative ring ([59], (10.10)).

(g) Let $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$ be a fixed but arbitrary infinite sequence of integers, where each of a_n is greater than 1. Consider the group $\mathbb{R} \times \Delta_{\boldsymbol{a}}$. Let B be the subgroup of the group $\mathbb{R} \times \Delta_{\boldsymbol{a}}$ of the form $B = \{(n, n\boldsymbol{u})\}_{n=-\infty}^{\infty}$, where $\boldsymbol{u} = (1, 0, \dots, 0, \dots)$. The factor group $\Sigma_{\boldsymbol{a}} = (\mathbb{R} \times \Delta_{\boldsymbol{a}})/B$ is called an *$\boldsymbol{a}$-adic solenoid* ([59], (10.12)). In the particular case where $\Delta_{\boldsymbol{a}} = \Delta_p$, the factor group $(\mathbb{R} \times \Delta_p)/B$ is called a *p-adic solenoid* and is denoted by Σ_p. The group $\Sigma_{\boldsymbol{a}}$ is compact and connected ([59, (10.13)]), and has dimension 1 ([59], (24.28)).

(h) $\mathbb{Q}$: the additive group of rational numbers with the discrete topology.

1.3 Torsion-free groups of rank 1 ([51], §85). Let A be a torsion-free group, p be a prime number, and $a \in A$. The largest nonnegative integer k such that the equation $p^k x = a$ has a solution $x \in A$ is called the *p-height* of the element a, and is denoted by $h_p(a)$. If no such maximal nonnegative integer k exists, then we set $h_p(a) = \infty$. The sequence of p-heights

$$\chi(a) = (h_{p_1}(a), \dots, h_{p_n}(a), \dots),$$

where p_n is the nth prime number, is said to be the *characteristic of an element* of the element a. Every sequence $(k_1, \dots, k_n, \dots)$ of nonnegative integers and symbols ∞ is a characteristic. Namely, consider the subgroup A of the group of rational numbers $\mathbb{Q}$ generated by all elements $p_n^{-l_n}$ with $l_n \le k_n$ for all natural n. It is clear that the sequence $(k_1, \dots, k_n, \dots)$ is the characteristic of the element $1 \in A$. Two characteristics $(k_1, \dots, k_n, \dots)$ and $(m_1, \dots, m_n, \dots)$ are considered as equivalent if $k_n \ne m_n$ holds only for a finite number of n such that in case $k_n \ne m_n$ both k_n and m_n are finite. All characteristics fall into disjoint classes of equivalent characteristics. These classes are called *types*. We represent a type $\mathbf{t}$ by a characteristic of this class.

A torsion-free group A in which all the nonzero elements are of the same type $\mathbf{t}$ is called *homogeneous*, and the type $\mathbf{t}(A)$ common to all nonzero elements of A is called the *type of* A. Every subgroup B of the group $\mathbb{Q}$ is homogeneous. Namely, if $a \in B$ and $\chi(a) = (k_1, \dots, k_n, \dots)$, then B is isomorphic to the subgroup A described above. Every torsion-free group of rank 1 is homogeneous and isomorphic to some subgroup of the group $\mathbb{Q}$. Two torsion-free groups of rank 1 are isomorphic if and only if they are of the same type ([51], Theorem 85.1). Thus there is a one-to-one correspondence between torsion-free groups of rank 1 and types. We note as examples that $\mathbf{t}(\mathbb{Z}) = (0, \dots, 0, \dots)$, $\mathbf{t}(\mathbb{Q}) = (\infty, \dots, \infty, \dots)$. We also note that $(\infty, 0, \dots, 0, \dots)$ is the type of the subgroup of dyadic rational numbers, and $(1, \dots, 1, \dots)$ is the type of the subgroup of the group $\mathbb{Q}$ with square-free denominators.

1.4 The character group. A *character* of a locally compact Abelian group X is a continuous homomorphism of X into the circle group $\mathbb{T}$. We denote the set of all characters of the group X by X^*. Put $Y = X^*$. The value of the character $y \in Y$ at the point $x \in X$ is denoted by (x, y). Obviously, Y is an Abelian group with respect to pointwise multiplication. For every compact set F of X, and every $\varepsilon > 0$, let $P(F, \varepsilon)$ be the set $\{y \in Y : |(x, y) - 1| < \varepsilon \text{ for all } x \in F\}$. With all sets $P(F, \varepsilon)$ taken as an open basis at zero in Y, Y is a locally compact Abelian group ([59], (23.15)). The group Y is called the *character group* of X. Let τ be the mapping $\tau\colon X \mapsto Y^*$ defined by $(y, \tau x) = (x, y)$. The following assertion is a central theorem in abstract harmonic analysis.

Pontryagin's duality theorem 1.5 ([59], (24.8)). *The mapping τ is a topological isomorphism from the group X into Y^*.*

1.6 Duality properties of groups X and Y. According to the Pontryagin duality theorem, every topological or algebraic property of a locally compact Abelian group can be described in the terms of topological or algebraic properties of its character group. The following theorems hold.

1. *A locally compact Abelian group is compact if and only if its character group is discrete* ([59], (23.17)).
2. *A compact Abelian group is connected if and only if its character group is torsion-free* ([59], (24.25)).
3. *For a compact connected Abelian group X the dimension of X is equal to the rank $r(Y)$ of its character group. If a connected group X is second countable, then* $\dim X \le \aleph_0$. ([59], (24.25), (24.28)).
4. *A compact Abelian group is totally disconnected if and only if its character group is a torsion group* ([59], (24.26)).

1.7 Character groups of direct products. We can compute the character group of a direct product of groups knowing the character groups of factors. The following theorems hold.

1. *Let X_j, $j = 1, 2, \dots, n$, be locally compact Abelian groups, and let $Y_j = X_j^*$. Then*
$$(X_1 \times \cdots \times X_n)^* \cong Y_1 \times \cdots \times Y_n$$
([59], (23.18)).

2. *Let $\{K_\iota : \iota \in \mathcal{I}\}$ be a nonvoid family of compact Abelian groups, and let $K_\iota^* = D_\iota$. Then*

$$\Big(\mathop{\mathbf{P}}_{\iota\in\mathcal{I}} K_\iota\Big)^* \cong \mathop{\mathbf{P}^*}_{\iota\in\mathcal{I}} D_\iota.$$

If $\{D_\iota : \iota \in \mathcal{I}\}$ is a nonvoid family of discrete Abelian groups, and let $D_\iota^* = K_\iota$, then

$$\Big(\mathop{\mathbf{P}^*}_{\iota\in\mathcal{I}} D_\iota\Big)^* \cong \mathop{\mathbf{P}}_{\iota\in\mathcal{I}} K_\iota,$$

([59], (23.21)).

1.8 The annihilator. Let X be a locally compact Abelian group, Y be its character group, and B be a nonvoid subset of X. Put

$$A(Y, B) = \{y \in Y : (x, y) = 1 \text{ for all } x \in B\}.$$

The set $A(Y, B)$ is called the *annihilator* of B in Y.

1.9 Some corollaries from the Pontryagin duality theorem. The following theorems hold.

1. *Let G be a closed subgroup of a group X. Then $G = A(X, A(Y, G))$* ([59], § 24.10).
2. *Let G be a closed subgroup of a group X. The character group of the group G is topologically isomorphic to the factor group $Y/A(Y, G)$. Every continuous character of G has the form $x \mapsto (x, y)$ for some $y \in Y$. Two characters y_1 and y_2 of Y define the same character of the subgroup G if and only if $y_1 - y_2 \in A(Y, G)$. The character group of the factor group X/G is topologically isomorphic to the group $A(Y, G)$* ([59], (24.11), (23.25)).
3. *The sets b_X and c_X are closed subgroups of a group X. The following equalities hold: $b_Y = A(Y, c_X)$, $c_X = A(X, b_Y)$* ([59], (24.17)).
4. *Let K be a compact subgroup of a group X. Then the annihilator $A(Y, K)$ is an open subgroup of Y. If H is an open subgroup of Y, then $A(X, H)$ is a compact subgroup of X* ([59], (23.29)).
5. *For any natural n we have $A\big(Y, X^{(n)}\big) = Y_{(n)}$, $A\big(Y, X_{(n)}\big) = \overline{Y^{(n)}}$* ([59], (24.22)).
6. *If a group X is connected, then $X^{(n)} = X$ for all natural n* ([59], (24.25)).

1.10 Some examples of character groups. Now we consider examples of character groups of some locally compact Abelian groups.

(a) $\mathbb{R}^* \cong \mathbb{R}$, $(x, y) = e^{ixy}$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.
(b) $\mathbb{Z}^* \cong \mathbb{T}$, $(n, z) = z^n$, where $n \in \mathbb{Z}$, $z \in \mathbb{T}$.
(c) $\mathbb{Z}(m)^* \cong \mathbb{Z}(m)$, $(k, l) = \exp\{2\pi i kl/m\}$, where $k \in \mathbb{Z}(m)$, $l \in \mathbb{Z}(m)$.
(d) $\Delta_p^* \cong \mathbb{Z}(p^\infty)$,

$$(\boldsymbol{x}, y) = \exp\Big\{(x_0 + x_1 p + \cdots + x_{n-1} p^{n-1})\tfrac{2\pi i l}{p^n}\Big\},$$

where

$$\boldsymbol{x} = (x_0, x_1, x_2, \dots) \in \Delta_p, \quad y = \exp\left\{\frac{2\pi i l}{p^n}\right\} \in \mathbb{Z}(p^\infty)$$

([59], (25.2)).

(e) Consider the group $\Sigma_{\boldsymbol{a}} = (\mathbb{R} \times \Delta_{\boldsymbol{a}})/B$, where $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$. Then $\Sigma_{\boldsymbol{a}}^* \cong H_{\boldsymbol{a}}$, where

$$H_{\boldsymbol{a}} = \left\{\frac{m}{a_0 a_1 \dots a_n} : n = 0, 1, \dots; m \in \mathbb{Z}\right\}$$

is a subgroup of the group $\mathbb{Q}$, and

$$(\boldsymbol{x}, y) = \exp\left\{(t - (x_0 + a_0 x_1 + \dots + a_0 a_1 \dots a_{n-1} x_n))\frac{2\pi i m}{a_0 a_1 \dots a_n}\right\},$$

where $\boldsymbol{x} = (t; x_0, x_1, x_2, \dots) + B \in \Sigma_{\boldsymbol{a}}$, $(t; x_0, x_1, x_2, \dots) \in \mathbb{R} \times \Delta_{\boldsymbol{a}}$, $y = \frac{m}{a_0 a_1 \dots a_n} \in H_{\boldsymbol{a}}$ ([59], (25.3)). We note that if $\boldsymbol{a} = (2, 3, 4, \dots)$, then $\Sigma_{\boldsymbol{a}}^* \cong \mathbb{Q}$.

1.11 Structure theorems for some classes of locally compact Abelian groups. The following theorems hold.

1. *A locally compact Abelian group X is topologically isomorphic to the group $\mathbb{R}^m \times G$, where $m \geq 0$ and G contains a compact open subgroup* ([59], (24.30)).

2. *A connected locally compact Abelian group X is topologically isomorphic to the group $\mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact connected group* ([59], (9.14)).

3. *Let X be a second countable connected locally compact Abelian group. Then the following statements are equivalent:*

(i) *the group X is locally connected;*

(ii) *the group X is arcwise connected;*

(iii) *the group X is topologically isomorphic to the group $\mathbb{R}^m \times \mathbb{T}^{\mathfrak{n}}$, where $m \geq 0$ and $\mathfrak{n} \leq \aleph_0$* ([1], (8.27)).

4. *A compact Abelian torsion-free group is topologically isomorphic to the group*

$$(\Sigma_{\boldsymbol{a}})^{\mathfrak{n}} \times \mathbf{P}_{p \in \mathcal{P}} \Delta_p^{\mathfrak{n}_p},$$

where $\boldsymbol{a} = (2, 3, 4, \dots)$ and $\mathfrak{n}$, $\mathfrak{n}_p$ are cardinal numbers ([59], (25.8)).

5. *Let X be a locally compact Abelian group such that every element of X different from zero has order p, where p is a fixed prime number. Then X is topologically isomorphic to the group*

$$\mathbb{Z}(p)^{\mathfrak{n}} \times \mathbb{Z}(p)^{\mathfrak{m}*},$$

where $\mathfrak{n}$ and $\mathfrak{m}$ are arbitrary cardinal numbers, $\mathbb{Z}(p)^{\mathfrak{n}}$ is considered in the product topology, and $\mathbb{Z}(p)^{\mathfrak{m}*}$ is considered in the discrete topology ([59], (25.29)).

1.12 Small subgroups in locally compact Abelian groups. The following theorems hold.

1. *Let X be a totally disconnected locally compact Abelian group. Then every neighbourhood of zero in X contains a compact open subgroup* ([59], (7.7)).
2. *Let X be a compact Abelian group. Then every neighbourhood of zero in X contains a closed subgroup G such that the factor group X/G is topologically isomorphic to the group $\mathbb{T}^m \times F$, where $m \geq 0$ and F is a finite Abelian group* ([59], (24.7)).

1.13 Adjoint homomorphisms. Let X_1 and X_2 be locally compact Abelian groups with character groups Y_1 and Y_2 respectively. Let $p\colon X_1 \mapsto X_2$ be a continuous homomorphism. For each $y_2 \in Y_2$ define the mapping $\tilde{p}\colon Y_2 \mapsto Y_1$ by the equality $(px_1, y_2) = (x_1, \tilde{p}y_2)$ for all $x_1 \in X_1$, $y_2 \in Y_2$. The mapping $\tilde{p}$ is a continuous homomorphism. It is called an *adjoint* of p. We list some properties of adjoint homomorphisms ([59], (24.41)).

(a) The homomorphism p satisfies the condition $p = \tilde{\tilde{p}}$.

(b) The homomorphism $\tilde{p}$ is a monomorphism if and only if the subgroup $p(X_1)$ is dense in X_2.

(c) The homomorphism $\tilde{p}$ is a topological isomorphism from the group Y_2 into Y_1 if and only if p is a topological isomorphism from the group X_1 into X_2.

(d) Let f_n be the homomorphism $f_n x = nx$ of X into X. Then the adjoint homomorphism $\widetilde{f_n}\colon Y \mapsto Y$ is the mapping $\widetilde{f_n} y = ny$.

1.14 Automorphism groups of some locally compact Abelian groups. Let X be a locally compact Abelian group, Y be its character group. Denote by $\mathrm{Aut}(X)$ the group of all topological automorphisms of the group X, i.e., the group of all mappings of X into X that are simultaneously algebraic automorphisms and homeomorphisms. Denote by I the identical automorphism of a group. It is possible to define a topology for the group $\mathrm{Aut}(X)$ in such a manner that $\mathrm{Aut}(X)$ becomes a topological group ([59], (26.3)]). In studying characterization problems, we do not need to consider the group $\mathrm{Aut}(X)$ as a topological group. Therefore, when we say that a group $\mathrm{Aut}(X)$ is isomorphic to some group, we keep in mind an algebraic rather than topological isomorphism, in spite of the fact that in the cases under consideration a topological isomorphism also holds. Let $\alpha \in \mathrm{Aut}(X)$. The mapping $\alpha \mapsto \tilde{\alpha}$ is an anti-isomorphism of the groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ ([59], (26.9)). A subgroup G of the group X is said to be *characteristic* if G is invariant under each automorphism $\alpha \in \mathrm{Aut}(X)$. The subgroups c_X, b_X, $X_{(n)}$ and $X^{(n)}$ are characteristic. We now consider examples of automorphism groups $\mathrm{Aut}(X)$ of some locally compact Abelian groups X.

(a) Every topological automorphism α of the group $\mathbb{R}$ is of the form $\alpha t = ct$, where $c \in \mathbb{R}$, $c \neq 0$, $t \in \mathbb{R}$. It follows from this that the group $\mathrm{Aut}(\mathbb{R})$ is isomorphic to the multiplicative group of all nonzero real numbers, i.e., $\mathrm{Aut}(\mathbb{R})$ is isomorphic to the group $\mathbb{R} \times \mathbb{Z}(2)$.

(b) The group $\mathbb{Z}$ admits just two automorphisms $\pm I$. This implies that the group $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to the group $\mathbb{Z}(2)$.

(c) Let p be an arbitrary prime number. The group $\operatorname{Aut}(\Delta_p)$ is isomorphic to the multiplicative group of all invertible elements of the ring Δ_p, i.e., $\operatorname{Aut}(\Delta_p)$ is isomorphic to the multiplicative group $\Delta_p^0 = \{\boldsymbol{c} = (c_0, c_1, c_2, \dots) \in \Delta_p : c_0 \neq 0\}$. To prove this we note that if $\boldsymbol{c} \in \Delta_p^0$ we can define the automorphism $\alpha_{\boldsymbol{c}} \in \operatorname{Aut}(\Delta_p)$ by the formula $\alpha_{\boldsymbol{c}}\boldsymbol{x} = \boldsymbol{c}\boldsymbol{x}$, $\boldsymbol{x} \in \Delta_p$. Let $\alpha \in \operatorname{Aut}(\Delta_p)$ be an arbitrary automorphism. Denote by $\boldsymbol{u}$ the element $(1, 0, 0, \dots) \in \Delta_p$. Since the subgroup generated by $\boldsymbol{u}$ is dense in Δ_p, the same is also true for the element $\alpha\boldsymbol{u}$. Hence $\alpha\boldsymbol{u} = \boldsymbol{c}$, where $\boldsymbol{c} \in \Delta_p^0$. It follows from this that $\alpha = \alpha_{\boldsymbol{c}}$ ([59], (26.18e)).

(d) To find the group $\operatorname{Aut}(\Sigma_{\boldsymbol{a}})$, where $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$, we note that $\Sigma_{\boldsymbol{a}}{}^* \cong H_{\boldsymbol{a}}$, where $H_{\boldsymbol{a}} = \left\{\frac{m}{a_0 a_1 \dots a_n} : n = 0, 1, \dots; m \in \mathbb{Z}\right\}$ is a subgroup of $\mathbb{Q}$ (see 1.10 (e)). It is easy to see that either the group $H_{\boldsymbol{a}}$ admits just two automorphisms $\pm I$, and hence the group $\operatorname{Aut}(H_{\boldsymbol{a}})$ is isomorphic to the group $\mathbb{Z}(2)$, or every automorphism α of the group $H_{\boldsymbol{a}}$ is of the form $\alpha r = \frac{m}{n} r$, $r \in H_{\boldsymbol{a}}$, where m, n are some mutually prime numbers. Furthermore $f_m, f_n \in \operatorname{Aut}(H_{\boldsymbol{a}})$, and at least one of the automorphisms f_m, f_n is different from $\pm I$. If the number of primes p such that $f_p \in \operatorname{Aut}(H_{\boldsymbol{a}})$ is finite and is equal to l, it is easily seen that the group $\operatorname{Aut}(H_{\boldsymbol{a}})$ is isomorphic to $\mathbb{Z}^l \times \mathbb{Z}(2)$. If this number is infinite, then the group $\operatorname{Aut}(H_{\boldsymbol{a}})$ is isomorphic to $\mathbb{Z}^{\aleph_0 *} \times \mathbb{Z}(2)$. Since the groups $\operatorname{Aut}(\Sigma_{\boldsymbol{a}})$ and $\operatorname{Aut}(H_{\boldsymbol{a}})$ are isomorphic, we find the group $\operatorname{Aut}(\Sigma_{\boldsymbol{a}})$. Moreover, since every automorphism $\alpha \in \operatorname{Aut}(H_{\boldsymbol{a}})$ is of the form $\alpha = f_m f_n^{-1}$, it results from 1.13 (d) that every automorphism $\tilde{\alpha} \in \operatorname{Aut}(\Sigma_{\boldsymbol{a}})$ is of the same form.

We note that if p_j is the jth prime number, then $f_{p_j} \in \operatorname{Aut}(\Sigma_{\boldsymbol{a}})$ if and only if the symbol ∞ stands at the jth place in $\mathbf{t}(H_{\boldsymbol{a}})$.

(e) The group $\operatorname{Aut}(\mathbb{T}^m)$ is isomorphic to the group of all integer-valued $(m \times m)$-matrices $A = (a_{ij})_{i,j=1}^m$ such that $\det A = \pm 1$. If $\alpha \in \operatorname{Aut}(\mathbb{T}^m)$ and $A = (a_{ij})_{i,j=1}^m$ is the corresponding matrix, then

$$\alpha(z_1, \dots, z_m) = (z_1^{a_{11}} z_2^{a_{21}} \dots z_m^{a_{m1}}, \dots, z_1^{a_{1m}} z_2^{a_{2m}} \dots z_m^{a_{mm}}),$$

where $(z_1, \dots, z_m) \in \mathbb{T}^m$ ([59], (26.18h)).

1.15 $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ as a subgroup of the infinite-dimensional torus $\mathbb{T}^{\aleph_0}$. It is well known that every second countable compact Abelian group X is topologically isomorphic to a compact subgroup G in the infinite-dimensional torus $\mathbb{T}^{\aleph_0}$. We will describe this subgroup G for an $\boldsymbol{a}$-adic solenoid $X = \Sigma_{\boldsymbol{a}}$. For the sake of simplicity, we consider the case when $\boldsymbol{a} = (p, p, \dots, p, \dots)$, where p is a prime number, i.e., we consider the case of a p-adic solenoid Σ_p. The corresponding reasoning can be carried out for an arbitrary $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$. Let $\boldsymbol{u} = (1, 0, \dots, 0, \dots) \in \Delta_p$, $B = \{(n, n\boldsymbol{u})\}_{n=-\infty}^{\infty} \subset \mathbb{R} \times \Delta_p$, and $\Sigma_p = (\mathbb{R} \times \Delta_p)/B$ be the corresponding p-adic solenoid. Consider the mapping $\tau\colon \mathbb{R} \times \Delta_p \mapsto \mathbb{T}^{\aleph_0}$, defined by

$$\tau(t, \boldsymbol{x}) = (z_1, z_2, \dots, z_n, \dots), \ t \in \mathbb{R}, \quad \boldsymbol{x} = (x_0, x_1, \dots, x_n, \dots) \in \Delta_p, \tag{1.1}$$

$$z_n = \exp\left\{\tfrac{2\pi i}{p^{n-1}}\Big(t - \sum_{k=0}^{n-1} x_k p^k\Big)\right\}. \tag{1.2}$$

It is not difficult to verify that τ is a continuous homomorphism, $\operatorname{Ker}\tau = B$ and $G = \tau(\mathbb{R} \times \Delta_p) = \{z = (z_1, z_2, \ldots, z_n, \ldots) \in \mathbb{T}^{\aleph_0} : z_k^p = z_{k-1},\ k \geq 2\}$ is a closed subgroup of the infinite-dimensional torus $\mathbb{T}^{\aleph_0}$. It follows from ([59], (5.27)) and ([59], (5.29)) that $G \cong \Sigma_p$.

The consideration of the p-adic solenoid Σ_p as the subgroup

$$G = \{z = (z_1, z_2, \ldots, z_n, \ldots) \in \mathbb{T}^{\aleph_0} : z_k^p = z_{k-1},\ k \geq 2\}$$

allows us to verify easily the following:

(a) if $h = \frac{m}{p^n}$ is a character of the group Σ_p, then $(z, h) = z_{n+1}^m$, where $z = (z_1, z_2, \ldots, z_n, \ldots) \in G$;

(b) the automorphism f_p operates on G by the formula $f_p z = (z_1^p, z_2^p, \ldots, z_n^p, \ldots)$, $z = (z_1, z_2, \ldots, z_n, \ldots) \in G$;

(c) $\tau(\mathbb{R}) = \{(e^{2\pi i t}, e^{2\pi i t/p}, e^{2\pi i t/p^2}, \ldots), t \in \mathbb{R}\}$ is the dense one-parameter subgroup of G.

1.16 Closed subgroups of the group $\mathbb{R}^m$. *Let G be a closed subgroup of the group $\mathbb{R}^m$. Then G is topologically isomorphic to the group $\mathbb{R}^p \times \mathbb{Z}^q$, where $p + q \leq m$.*

1.17 Subgroups $\mathbb{T}^{\mathfrak{n}}$ as direct factors. Let X be a locally compact Abelian group, Y be its character group, $\mathfrak{n}$ be a nonzero cardinal number. Then the following theorem holds.

1. *If X contains a subgroup G, and G is topologically isomorphic to $\mathbb{T}^{\mathfrak{n}}$, then G is a topological direct factor of X* ([59], (25.31)).

Taking into account Theorem 1.9.2 and the topological isomorphism $(\mathbb{T}^{\mathfrak{n}})^* \cong \mathbb{Z}^{\mathfrak{n}*}$, this theorem can be reformulated as follows.

2. *If Y contains a closed subgroup H such that the factor group Y/H is topologically isomorphic to the group $\mathbb{Z}^{\mathfrak{n}*}$, then the group Y is topologically isomorphic to the group $H \times \mathbb{Z}^{\mathfrak{n}*}$.*

Now we come to the results concerning algebraic group theory. For this reason in the rest of the section we assume that all groups under consideration are Abelian and discrete.

1.18 Some more definitions. A group X is said to be *reduced*, if X contains no nonzero divisible subgroups. A subgroup G of a group X is called *pure* if for any natural n and $g \in G$ the solvability of the equation $nx = g$ in X implies its solvability in G. A torsion group X is called *p-primary*, if the order of every element of X is a power of p.

1.19 Decomposition theorems. Let X be an Abelian group. The following theorems hold.

1. *Let X be a torsion group. For each prime number p let X_p be the subgroup of X consisting of elements whose order is a power of p. Then*

$$X = \mathop{\mathbf{P}^*}_{p\in\mathcal{P}} X_p$$

([50], Theorem 8.4). The subgroups X_p are called *p-components* of X.

2. *The group X can be decomposed into a direct product $X = D \times N$, where D is the divisible group, and N is a reduced group. The subgroup D is uniquely determined, and the subgroup N is uniquely determined up to isomorphism* ([50], Theorem 21.3).

3. *Let X be a divisible group. Then X is isomorphic to a weak direct product*

$$\mathbb{Q}^{\mathfrak{n}_0 *} \times \mathop{\mathbf{P}^*}_{p\in\mathcal{P}} \mathbb{Z}(p^\infty)^{\mathfrak{n}_p *},$$

where $\mathfrak{n}_0$ and $\mathfrak{n}_p$ are cardinal numbers ([59], § A.14).

4. *Let X be a group generated by elements $x_1, \dots, x_n$. Then*

$$X = X_1 \times \cdots \times X_n,$$

where X_j is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}(n_j)$ ([59], § A.27).

1.20 Reduced p-primary Abelian groups. Let A be a reduced p-primary Abelian group. Put

$$A^1 = \bigcap_{n=1}^{\infty} A^{(p^n)}, \quad A^{\sigma+1} = (A^\sigma)^1, \quad A^\sigma = \bigcap_{\rho<\sigma} A^\rho \quad \text{if } \sigma \text{ is a limit ordinal number.}$$

Let τ be the least ordinal number for which $A^\tau = \{0\}$. Then the totally ordered sequence of subgroups

$$A = A^0 \supset A^1 \supset \cdots \supset A^\sigma \supset \cdots \supset A^\tau = \{0\}$$

is defined. The factor group $A_\sigma = A^\sigma / A^{\sigma+1}$ is called the σth *Ulm factor* of the group A. The totally ordered sequence of Ulm factors

$$A_0, A_1, \dots, A_\sigma, \dots \quad (\sigma < \tau)$$

is called the Ulm sequence of the group A, and τ is called the *Ulm type* of the group A. For each ordinal number σ define the subgroup $A^{(p^\sigma)}$ as follows:

$$A^{(p^0)} = A, \quad A^{(p^{\sigma+1})} = \left(A^{(p^\sigma)}\right)^{(p)}, \quad A^{(p^\sigma)} = \bigcap_{\rho<\sigma} A^{(p^\rho)} \quad \text{if } \sigma \text{ is a limit ordinal number.}$$

Consider the factor group $\left(A^{(p^\sigma)}\right)_{(p)} / \left(A^{(p^{\sigma+1})}\right)_{(p)}$. Its rank is called the σth *Ulm–Kaplansky invariant* of the group A. We note that $A^\sigma = A^{(p^{\omega\sigma})}$.

The following two theorems are the base of the theory of countable reduced p-primary Abelian groups.

1. *A countable reduced p-primary Abelian group A of the Ulm type τ and with the Ulm sequence A_σ $(0 \le \sigma < \tau)$ exists if and only if*
1) *τ is a countable ordinal number;*
2) *every group A_σ is a weak direct product of cyclic p-primary groups; moreover if $\sigma + 1 < \tau$, then A_σ contains elements of arbitrarily large order* ([51], Theorem 76.2).

2. *Two countable reduced p-primary Abelian groups A and C are isomorphic if and only if they have the same Ulm type τ, and for every ordinal number $\sigma < \tau$ their Ulm factors A_σ and C_σ are isomorphic.*
The latter holds if and only if the Ulm–Kaplansky invariants of the groups A and C coincide ([51], Theorem 77.3).

It should be noted that if A is a countable group, then the number of cyclic direct factors of order p^n in the decomposition of A_σ coincides with the $(\omega\sigma + n - 1)$-th Ulm–Kaplansky invariant of the group A.

2 Probability distributions on locally compact Abelian groups

We present in this section results of probability distributions on locally compact Abelian groups. We use for references the books by Grenander ([57]) and Parthasarathy ([89]). We agree to assume, unless otherwise specified, that all groups considered in this section are second countable, in spite of the fact that some of the statements below are true without this restriction.

2.1 Main definitions and notation. Let X be a topological Abelian group, let $\mathfrak{B}(X)$ be the σ-algebra of Borel sets, i.e., the smallest σ-algebra of subsets of X which contains all the open subsets of X. By a *measure* on the group X we understand a nonnegative countable additive set function defined on $\mathfrak{B}(X)$.

The difference of two measures is called a *signed measure*. A measure μ is called a *distribution* if $\mu(X) = 1$. Denote by $\mathrm{M}^1(X)$ the set of all distributions on X.

Denote by $\sigma(\mu)$ the *support* of a distribution μ, i.e., the smallest closed subset $A \subset X$ such that $\mu(A) = \mu(X)$. Let $\mu, \nu \in \mathrm{M}^1(X)$. We define their *convolution* $\mu * \nu \in \mathrm{M}^1(X)$ by the formula

$$(\mu * \nu)(B) = \int_X \mu(B - x)\, d\nu(x), \quad B \in \mathfrak{B}(X).$$

The set $\mathrm{M}^1(X)$ is a semigroup with respect to the convolution. Denote by μ^{*n} the convolution of n distributions each of which is equal to μ. If $\mu = \mu_1 * \mu_2$, then distributions μ_j are called *factors* of the distribution μ. For $\mu \in \mathrm{M}^1(X)$ define the distribution $\bar{\mu} \in \mathrm{M}^1(X)$ by the formula $\bar{\mu}(B) = \mu(-B)$ for all $B \in \mathfrak{B}(X)$.

We say that a *distribution μ is concentrated on a set* $A \in \mathfrak{B}(X)$ if $\mu(B) = 0$ for any $B \in \mathfrak{B}(X)$ such that $B \cap A = \emptyset$. In general the set A does not need to be closed, and A

is not uniquely determined. Let $x \in X$. Denote by E_x the *degenerate distribution* concentrated at the point $x \in X$, and by $D(X)$ the set of all degenerate distributions on X. The convolution $\mu * E_x$ of a distribution μ with a degenerate distribution is called a *shift of a distribution* μ.

Let $(\Omega, \mathfrak{A}, P)$ be a probability space, where Ω is a set, $\mathfrak{A}$ is a σ-algebra of subsets of Ω, and $P(A)$ is a probability measure defined on $\mathfrak{A}$. By a *random variable* on $(\Omega, \mathfrak{A}, P)$ with values in X we understand a function $\xi(\omega)$ defined on Ω with values in X, and such that $\xi^{-1}(B) \in \mathfrak{A}$ for any Borel subset $B \subset X$. The random variable ξ defines a distribution μ_ξ on $\mathfrak{B}(X)$ by the formula

$$\mu_\xi(B) = P\{\omega \in \Omega : \xi(\omega) \in B,\ B \in \mathfrak{B}(X)\}.$$

Two random variables ξ and η with values in X are called *independent* if the equality

$$P\{\omega : \xi(\omega) \in A,\ \eta(\omega) \in B\} = P\{\omega : \xi(\omega) \in A\}P\{\omega : \eta(\omega) \in B\}$$

is fulfilled for all $A, B \in \mathfrak{B}(X)$. If random variables ξ and η are independent, then

$$\mu_{\xi+\eta} = \mu_\xi * \mu_\eta. \tag{2.1}$$

The following simple statement proves to be useful.

Proposition 2.2. *Let X be a topological Abelian group, G be a Borel subgroup of X, $\mu \in \mathrm{M}^1(G)$, $\mu = \mu_1 * \mu_2$, where $\mu_j \in \mathrm{M}^1(X)$. Then the distributions μ_j can be replaced by their shifts μ_j' in such a manner that $\mu = \mu_1' * \mu_2'$ and $\mu_j' \in \mathrm{M}^1(G)$.*

Proof. We deduce from the equality

$$1 = \mu(G) = \int_X \mu_1(G - x) d\mu_2(x)$$

that $\mu_2\{x \in X : \mu_1(G - x) = 1\} = 1$. Hence $\mu_1(G - x_1) = 1$ for some $x_1 \in X$, i.e., the distribution μ_1 is concentrated on the set $G - x_1$. This implies that $\mu_1' = \mu_1 * E_{x_1} \in \mathrm{M}^1(G)$. Put $\mu_2' = \mu_2 * E_{-x_1}$. Then $\mu = \mu_1' * \mu_2'$ and

$$1 = \mu(G) = \int_X \mu_2'(G - x)\, d\mu_1'(x) = \int_G \mu_2'(G - x)\, d\mu_1'(x).$$

It follows from this that $\mu_1'\{x \in G : \mu_2'(G - x) = 1\} = 1$. Therefore $\mu_2'(G - x_2) = 1$ for some $x_2 \in G$. Since G is a group, we have $\mu_2'(G) = 1$, i.e., $\mu_2' \in \mathrm{M}^1(G)$. □

We also need the following general statement.

Suslin's theorem 2.3 ([69], §39, IV). *Let X_1 be a separable complete metric space, X_2 be a metric space, $p\colon X_1 \mapsto X_2$ be a continuous one-to-one mapping. If B is a Borel set in X_1, then $p(B)$ is a Borel set in X_2.*

Proposition 2.4. *Let X_1 and X_2 be topological Abelian groups, $p\colon X_1 \mapsto X_2$ be an algebraic isomorphism with the property that the images and preimages of Borel sets under the mapping p are Borel. Then p generates an isomorphism of the semigroups $\mathrm{M}^1(X_1)$ and $\mathrm{M}^1(X_2)$ by*

$$\text{(i)} \quad p(\mu)(B) = \mu(p^{-1}(B)),$$

where $\mu \in \mathrm{M}^1(X_1)$, $B \in \mathfrak{B}(X_2)$ (we keep the notation p for this isomorphism).

The proof of this statement consists of some standard verifications. The statement below follows directly from the Suslin theorem and Proposition 2.4.

Corollary 2.5. *Let X_1 be a complete topological Abelian metric group, X_2 a topological Abelian metric group, and let $p\colon X_1 \mapsto X_2$ be a continuous monomorphism. Then p generates an isomorphism of the semigroups $\mathrm{M}^1(X_1)$ and $\mathrm{M}^1(p(X_1))$ by formula 2.4 (i).*

2.6 Topology on $\mathbf{M}^1(X)$. Let X be a topological Abelian group. We introduce a weak topology in $\mathrm{M}^1(X)$ as follows: a sequence of distributions $\mu_n \in \mathrm{M}^1(X)$ converges to a distribution $\mu \in \mathrm{M}^1(X)$ ($\mu_n \Rightarrow \mu$) if

$$\int_X f(x)\, d\mu_n(x) \to \int_X f(x)\, d\mu(x)$$

for every bounded continuous function $f(x)$ on X.

2.7 Characteristic functions. Let X be a locally compact Abelian group, Y be its character group. For every $\mu \in \mathrm{M}^1(X)$ the *characteristic function* (Fourier transform) of μ is defined by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x).$$

A function $f(y)$ on the group Y is called *characteristic* if $f(y) = \hat{\mu}(y)$ for a distribution $\mu \in \mathrm{M}^1(X)$. We note that if ξ is a random variable with values in X and distribution μ, then the characteristic function of μ is the expectation

$$\hat{\mu}(y) = \mathbf{E}[(\xi, y)]. \tag{2.2}$$

The *characteristic function of a random variable* is the characteristic function of its distribution. If ξ and η are random variables with values in X, then ξ and η are independent if and only if the equality

$$\mathbf{E}[(\xi, u)(\eta, v)] = \mathbf{E}[(\xi, u)]\mathbf{E}[(\eta, v)]$$

holds for all $u, v \in Y$.

Let $\mu \in \mathrm{M}^1(X)$. The characteristic function $\hat{\mu}(y)$ has the following properties ([57], § 3.3):

(a) $|\hat{\mu}(y)| \leq \hat{\mu}(0) = 1$;
(b) μ is uniquely defined by the function $\hat{\mu}(y)$;
(c) $(\widehat{\mu * \nu})(y) = \hat{\mu}(y)\hat{\nu}(y)$, $\mu, \nu \in \mathrm{M}^1(X)$;
(d) $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)} = \hat{\mu}(-y)$;
(e) if H is a closed subgroup of the group Y and $|\hat{\mu}(y)| = 1$ for all $y \in H$, then there exists an element $x \in X$ such that $\hat{\mu}(y) = (x, y)$ for all $y \in H$;
(f) $\hat{\mu}(y)$ is a uniformly continuous function;
(g) the inequality

$$|\hat{\mu}(y_1) - \hat{\mu}(y_2)|^2 \leq 2(1 - \mathrm{Re}\, \hat{\mu}(y_1 - y_2))$$

holds for all $y_1, y_2 \in Y$;
(h) let $\mu_n \in \mathrm{M}^1(X)$; $\mu_n \Rightarrow \mu$ if and only if $\hat{\mu}_n(y) \to \hat{\mu}(y)$ uniformly on every compact subset of Y;
(i) let $\mu_n \in \mathrm{M}^1(X)$, and let the sequence $\hat{\mu}_n(y)$ converge to a limit uniformly on every compact subset of Y; then there exists a distribution $\mu \in \mathrm{M}^1(X)$ such that $\hat{\mu}_n(y) \to \hat{\mu}(y)$ and $\mu_n \Rightarrow \mu$.

We note that properties (b)–(d) and (f) are also fulfilled for an arbitrary signed measure.

2.8 Positive definite functions. Let Y be an arbitrary Abelian group. A function $f(y)$ defined on Y is said to be *positive definite* if, for any natural n, any complex numbers $z_1, \dots, z_n$, and any points $y_1, \dots, y_n \in Y$, the inequality

$$\text{(i)} \qquad \sum_{i,j=1}^{n} f(y_i - y_j) z_i \bar{z}_j \geq 0$$

holds.

Bochner's theorem 2.9 ([60], (33.3)). *Let X be a locally compact Abelian group, Y be its character group. A function $f(y)$ on the group Y is the characteristic function of a distribution $\mu \in \mathrm{M}^1(X)$ if and only if the following conditions hold:* (i) *$f(y)$ is continuous;* (ii) *$f(y)$ is positive definite;* (iii) $f(0) = 1$.

Thanks to the Bochner theorem we can not make a difference between characteristic functions and normalized continuous positive definite functions.

Proposition 2.10. *Let X_1 and X_2 be locally compact Abelian groups with character groups Y_1 and Y_2, respectively. Let $p\colon X_1 \mapsto X_2$ be a continuous homomorphism generating the mapping $p\colon \mathrm{M}^1(X_1) \mapsto \mathrm{M}^1(X_2)$ by formula 2.4* (i). *If $\mu \in \mathrm{M}^1(X_1)$, then the characteristic function of the distribution $p(\mu)$ is of the form $\widehat{p(\mu)}(y_2) = \hat{\mu}(\tilde{p} y_2)$, where $y_2 \in Y_2$, and $\tilde{p}\colon Y_2 \mapsto Y_1$ is the homomorphism adjoint of p.*

Corollary 2.11. *Let X be a locally compact Abelian group, Y be its character group, G be a closed subgroup of X, $H = A(Y, G)$, $p\colon X \mapsto X/G$ be the natural homomorphism, and $\mu \in \mathrm{M}^1(X)$. Then the restriction of the characteristic function $\hat\mu(y)$ to H is the characteristic function of the distribution $p(\mu) \in \mathrm{M}^1(X/G)$.*

The following statement is often used in construction of positive definite functions.

Proposition 2.12 ([60], (32.43)). *Let Y be an arbitrary Abelian group and H be a subgroup of Y. Let $f_0(y)$ be a positive definite function on H, and $f(y)$ be the function on Y such that*

$$f(y) = \begin{cases} f_0(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$$

Then $f(y)$ is a positive definite function.

Proposition 2.13. *Let X be a locally compact Abelian group, Y be its character group, and $\mu \in \mathrm{M}^1(X)$. Then the set*

$$E = \{y \in Y : \hat\mu(y) = 1\}$$

is a closed subgroup of Y, the characteristic function $\hat\mu(y)$ is E-invariant, i.e., $\hat\mu(y)$ takes a constant value on each coset of the group Y with respect to E, and $\sigma(\mu) \subset A(X, E)$.

Proof. It is obvious that the inequality

$$1 - \mathrm{Re}(x, y_1 + y_2) \le 2[(1 - \mathrm{Re}(x, y_1)) + (1 - \mathrm{Re}(x, y_1))]$$

holds for all $x \in X$, $y_1, y_2 \in Y$. This implies the inequality

$$1 - \mathrm{Re}\,\hat\mu(y_1 + y_2) \le 2[(1 - \mathrm{Re}\,\hat\mu(y_1)) + (1 - \mathrm{Re}\,\hat\mu(y_1))]. \tag{2.3}$$

It results from (2.3) that E is a subgroup. Obviously, E is closed. Inequality 2.7 (g) implies that if $y_1 - y_2 \in E$, then $\hat\mu(y_1) = \hat\mu(y_2)$, i.e., the function $\hat\mu(y)$ is E-invariant. Put $H = Y/E$. By Theorem 1.9.2, $H^* \cong A(X, E)$. As has been stated above, the function $\hat\mu(y)$ can be considered as a function $f([y])$ on H. The function $f([y])$ is continuous, positive definite, and $f(0) = 1$. By Bochner's theorem $f([y]) = \hat\lambda([y])$, $\lambda \in \mathrm{M}^1(A(X, E))$. We have

$$\hat\mu(y) = \int\limits_X (x, y)d\mu(x) = f([y]) = \int\limits_{A(X,E)} (x, [y])d\lambda(x) = \int\limits_X (x, y)d\lambda(x) = \hat\lambda(y).$$

It follows from 2.7 (b) that $\mu = \lambda$. □

2.14 Haar measure and idempotent distributions. Let X be a locally compact Abelian group, Y be its character group. There exists a measure m_X on the group X with the following properties:

(a) $m_X(B + x) = m_X(B)$ for all $B \in \mathfrak{B}(X)$, $x \in X$;

(b) $m_X(-B) = m_X(B)$ for all $B \in \mathfrak{B}(X)$.

The measure m_X is called a *Haar measure*. If X is a compact group, then $m_X(X) < \infty$. We assume that in this case $m_X \in \mathrm{M}^1(X)$ ([59], Chapter 4).

Let K be a compact subgroup of X. We note that the characteristic function of the distribution m_K is of the form

$$\text{(i)} \quad \widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in A(Y, K), \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

A distribution $\mu \in \mathrm{M}^1(X)$ is said to be *idempotent* if $\mu^{*2} = \mu * E_x$ for some $x \in X$. The set $I(X)$ of all idempotent distributions on the group X coincides with the set of all shifts of the Haar distributions m_K of compact subgroups K of X. Indeed, let $\mu^{*2} = \mu * E_x$. Put $\lambda = \mu * E_{-x}$. This implies that $\lambda^{*2} = \lambda$, hence by 2.7 (c) $\hat{\lambda}^2(y) = \hat{\lambda}(y)$. It follows from this that either $\hat{\lambda}(y) = 0$ or $\hat{\lambda}(y) = 1$. Consider the set $E = \{y \in Y : \hat{\lambda}(y) = 1\}$. Put $K = A(X, E)$. By Theorem 1.9.1, $E = A(Y, K)$. Taking into account 2.7 (b) we obtain that $\lambda = m_K$. It should be also noted that $D(X) \subset I(X)$.

2.15 Infinitely divisible distributions. Let X be a locally compact Abelian group, Y be its character group. A distribution μ on the group X is said to be *infinitely divisible* if for each natural n there are an element $x_n \in X$ and a distribution $\mu_n \in \mathrm{M}^1(X)$ such that $\mu = \mu_n^{*n} * E_{x_n}$.

We give the examples of infinitely divisible distributions:

(i) idempotent distributions, in particular, degenerate distributions;

(ii) the generalized Poisson distribution $e(F)$ associated with a finite measure F,

$$e(F) = \exp\{-F(X)\}(E_0 + F + F^{*2}/2! + \cdots + F^{*n}/n! + \cdots).$$

We note some properties of infinitely divisible distributions:

(a) the characteristic function $\hat{\mu}(y)$ of an infinitely divisible distribution μ vanishes if and only if μ has a nondegenerate idempotent factor ([89], Theorem 4.2);

(b) if μ is an infinitely divisible distribution, then the set $N = \{y \in Y : \hat{\mu}(y) \neq 0\}$ is an open subgroup of Y ([89], Lemma 5.4);

(c) the set of infinitely divisible distributions is a closed subsemigroup in $\mathrm{M}^1(X)$ ([89], Theorem 4.1).

2.16 The Lévy–Khinchin formula. Let X be a locally compact Abelian group, Y be its character group. The Lévy–Khinchin formula is a assertion which gives a representation of the characteristic function of an infinitely divisible distribution on X. To formulate it we need the following statement.

There exists a function $g(x, y)$ on the product $X \times Y$ with the following properties:

(a) $g(x, y)$ is continuous in two variables x and y;

(b) $\sup_{x\in X} \sup_{y\in K} |g(x, y)| < \infty$ for every compact subset $K \subset Y$;

(c) $g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$ for all $x \in X,\ y_1, y_2 \in Y$, and $g(-x, y) = -g(x, y)$;

(d) if K is an arbitrary compact in Y, then there is a neighbourhood of zero U_K in X such that $(x, y) = \exp\{ig(x, y)\}$ for all $x \in U_K$ and $y \in K$;

(e) if K is an arbitrary compact in Y, then $g(x, y)$ converges to zero uniformly in $y \in K$ as x tends to zero of the group X ([89], Lemma 5.3).

We now can get the group analogue of the Lévy–Khinchin formula.

Let X be a locally compact Abelian group, Y be its character group. If μ is an infinitely divisible distribution on X without idempotent factors, then the characteristic function $\hat{\mu}(y)$ has the representation

$$\text{(i)} \quad \hat{\mu}(y) = (x_0, y)\exp\left\{ \int_{X\setminus\{0\}} [(x, y) - 1 - ig(x, y)]\, dF(x) - \varphi(y)\right\},$$

where x_0 is a fixed point of X, $g(x, y)$ is a function on $X \times Y$ which is independent of μ and has the properties mentioned above, F is a σ-finite measure with finite mass outside every neighbourhood of zero in X which satisfies the condition

$$\int_{X\setminus\{0\}} [1 - \mathrm{Re}(x, y)]\, dF(x) < \infty \quad \textit{for every } y \in Y,$$

and $\varphi(y)$ is a nonnegative continuous function satisfying the equation

$$\text{(ii)} \quad \varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.$$

Conversely, any function of type (i) *is the characteristic function of an infinitely divisible distribution on the group X without idempotent factors* ([89], Theorem 7.1).

Representation (i) is called the *Lévy–Khinchin formula*. Measure F is called the Lévy measure of the infinitely divisible distribution μ.

If μ is an infinitely divisible distribution without idempotent factors, we say that μ has the representation (x_0, F, φ), where x_0, F, and φ are as in (i). If the characteristic function of an infinitely divisible distribution μ without idempotent factors has two representations (x_1, F_1, φ_1) and (x_2, F_2, φ_2), then $\varphi_1(y) = \varphi_2(y)$, $y \in Y$, and $F_1 = F_2$ on the complement of the subgroup b_X. Representation (i) is unique if and only if either $b_X = \{0\}$ or $b_X = X_{(2)}$ ([89], Chapter IV, § 8).

The following statement allows us in some cases to reduce the study of arbitrary infinitely divisible distributions to the study of an infinitely divisible distribution without idempotent factors.

Theorem 2.17 ([89], Theorem 7.2). *Let X be a locally compact Abelian group, μ be an infinitely divisible distribution on X. Then $\mu = m_K * \nu$, where K is a compact subgroup of X, and ν is an infinitely divisible distribution without idempotent factors.*

Cramér's theorem 2.18 ([74], Theorem 6.3.2). *Let γ be a Gaussian distribution in the space $\mathbb{R}^m$, and $\gamma = \gamma_1 * \gamma_2$, where $\gamma_j \in \mathrm{M}^1(\mathbb{R}^m)$. Then γ_j are also Gaussian distributions.*

Theorem 2.19 ([74], Chapter VI, § 1). *Let $F(t)$, $t \in \mathbb{R}^m$, be a characteristic function, and $\Phi(t)$, $t \in \mathbb{R}^m$, be the restriction to $\mathbb{R}^m$ an entire function $\Phi(z)$, $z \in \mathbb{C}^m$. Assume that U is a neighbourhood of zero in $\mathbb{R}^m$ and*

$$\text{(i)} \quad F(t) = \Phi(t), \quad t \in U.$$

Then the function $F(t)$ can be extended to $\mathbb{C}^m$ as an entire function, and (i) *holds for all $t \in \mathbb{R}^m$.*

Proposition 2.20. *Let G be a locally compact Abelian group, $X = \mathbb{R}^m \times G$. Denote by (s, h), $s \in \mathbb{R}^m$, $h \in H$ elements of the group $X^* \cong \mathbb{R}^m \times H$, where $H = G^*$. Let $\mu \in \mathrm{M}^1(X)$, and assume that $\hat{\mu}(s, 0)$ is an entire function in s. Then $\hat{\mu}(s, h)$ is an entire function in s for every fixed $h \in H$, and the representation*

$$\hat{\mu}(s, h) = \int\limits_X e^{i\langle t,s\rangle}(g, h)\, d\mu(x)$$

holds for all $s \in \mathbb{C}$, $h \in H$.

2.21 Conditional distribution. Let X be a topological Abelian group. Let ξ and η be random variables taking values in X. By the conditional distribution of the random variable η given ξ, we understand a function $P_{\eta|\xi}(x, B)$ defined on $X \times \mathfrak{B}(X)$ such that

(i) for every fixed $x \in X$ the function $P_{\eta|\xi}(x, B)$ is a distribution on X;

(ii) for each fixed $B \in \mathfrak{B}(X)$ and each $C \in \mathfrak{B}(X)$ the equality

$$P\{\omega \in \Omega : \xi(\omega) \in C, \eta(\omega) \in B\} = \int\limits_C P_{\eta|\xi}(x, B)\, d\mu_\xi(x)$$

holds. We deduce from the definition of the condition distribution of the random variable η given ξ that this distribution is symmetric, i.e.,

$$P_{\eta|\xi}(x, B) = P_{\eta|\xi}(x, -B)$$

if and only if the equality

$$P\{\omega \in \Omega : \xi(\omega) \in C, \eta(\omega) \in B\} = P\{\omega \in \Omega : \xi(\omega) \in C, \eta(\omega) \in -B\}$$

holds for all $B, C \in \mathfrak{B}(X)$.

Chapter II
Gaussian distributions on locally compact Abelian groups

Let X be a second countable locally compact Abelian group. This chapter is devoted to Gaussian distributions on X. We define Gaussian distribution on a group X and study its properties. We describe completely groups X on which the classical Cramér and Marcinkiewicz theorems can be extended. We also consider Gaussian distributions in the sense of Urbanik, i.e., such distributions on X which any character transforms into Gaussian distributions on the circle group $\mathbb{T}$. We describe completely groups X for which the class of Gaussian distributions coincides with the class of Gaussian distributions in the sense of Urbanik.

3 Properties of Gaussian distributions

Let X be a second countable locally compact Abelian group, Y be its character group. Following Parthasarathy, we define in this section the Gaussian distribution on the group X and study its properties.

Definition 3.1 ([89], Chapter 4, § 6). A distribution γ on a group X is called *Gaussian* if its characteristic function can be represented in the form

$$\text{(i)} \quad \hat{\gamma}(y) = (x, y)\exp\{-\varphi(y)\}, \quad y \in Y,$$

where $x \in X$, and $\varphi(y)$ is a continuous nonnegative function on the group Y satisfying equation 2.16 (ii).

Denote by $\Gamma(X)$ the set of Gaussian distributions on X. A Gaussian distribution is called *symmetric* if $x = 0$ in (i). Denote by $\Gamma^s(X)$ the set of symmetric Gaussian distributions on X.

3.2. Definition 3.1 of the Gaussian distribution for the groups $X = \mathbb{R}^k$ and $X = \mathbb{T}^k$ coincides with the classical one. The assertion with respect to $X = \mathbb{R}^k$ follows from the fact that indeed, if $X = \mathbb{R}^k$, then $Y \cong \mathbb{R}^k$, and a continuous nonnegative function $\varphi(y)$ on $\mathbb{R}^k$ satisfying equation 2.16 (ii) is of the form

$$\varphi(y) = \langle Ay, y\rangle, \quad y = (y_1, \dots, y_k) \in \mathbb{R}^k,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^k$, and A is a symmetric positive semidefinite matrix. In particular, if $\mu \in \Gamma(\mathbb{R})$, then the characteristic function $\hat{\mu}(s)$ can be written in the form

$$\hat{\mu}(s) = \exp\left\{ias - \tfrac{\sigma^2 s^2}{2}\right\}, \quad a \in \mathbb{R},\ \sigma \geq 0,\ s \in \mathbb{R}. \tag{3.1}$$

If in (3.1) $\sigma > 0$, then μ has density $\rho(t)$ with respect to the Lebesgue measure

$$\rho(t) = \frac{d\mu}{dt} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-a)^2}{2\sigma^2}}.$$

For the group $X = \mathbb{T}^k$ in accordance with the classical definition, γ is a Gaussian distribution on $\mathbb{T}^k$ if γ is an image of a Gaussian distribution μ on $\mathbb{R}^k$ under the natural homomorphism $p\colon \mathbb{R}^k \mapsto \mathbb{T}^k \cong \mathbb{R}^k/(2\pi\mathbb{Z})^k$. If the characteristic function of a Gaussian distribution μ is of the form

$$\hat{\mu}(s) = \exp\{i\langle t, s\rangle - \langle As, s\rangle\}, \quad s \in \mathbb{R}^k,$$

where $t \in \mathbb{R}^k$, and A is a symmetric positive semidefinite matrix, then by Proposition 2.10,

$$\hat{\gamma}(n) = \widehat{p(\mu)}(n) = \exp\{i\langle t, n\rangle - \langle An, n\rangle\}, \quad n = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k.$$

We see that γ is a Gaussian distribution in accordance with Definition 3.1. It is easy to see that the converse statement is also true (compare with Proposition 3.8). We note that if $\mu \in \Gamma(\mathbb{R})$ is a nondegenerate distribution and $\gamma = p(\mu) \in \Gamma(\mathbb{T})$, then γ also has density

$$\varrho(e^{it}) = \frac{\sqrt{2\pi}}{\sigma} \sum_{n=-\infty}^{\infty} \exp\left\{-\tfrac{1}{2\sigma^2}(t - a + 2\pi n)^2\right\} \tag{3.2}$$

with respect to the Haar distribution $m_{\mathbb{T}}$, and the characteristic function $\hat{\gamma}(n)$ is of the form

$$\hat{\gamma}(n) = \exp\left\{ian - \tfrac{\sigma^2 n^2}{2}\right\}, \quad n \in \mathbb{Z}.$$

Remark 3.3. A continuous function $\varphi(y)$ satisfying equation 2.16 (ii) is also called a square form on a group Y. If we know a square form $\varphi(y)$ we can construct a 2-additive function by the formula

$$\psi(u, v) = \frac{1}{2}[\varphi(u + v) - \varphi(u) - \varphi(v)], \quad u, v \in Y. \tag{3.3}$$

Obviously, the function $\psi(u, v)$ satisfies the conditions

$$\text{(i)} \qquad \psi(u, v) = \psi(v, u)$$
$$\text{(ii)} \quad \psi(u + v, w) = \psi(u, w) + \psi(v, w).$$

It is easy to see that if a function $\psi(u, v)$ satisfies conditions (i) and (ii), then the function

$$\varphi(y) = \psi(y, y) \tag{3.4}$$

satisfies equation 2.16 (ii).

Remark 3.4. Let $\varphi(y)$ be a continuous nonnegative function on a group Y satisfying equation 2.16 (ii), and let $f(y) = \exp\{-\varphi(y)\}$. Then $f(y)$ is a characteristic function, and hence $f(y)$ is the characteristic function of a Gaussian distribution on the group X. Indeed, let $y_1, \dots, y_k$ be fixed elements of Y. Consider the function $\varphi(n_1 y_1 + \dots + n_k y_k)$ as a function of integer-valued variables n_j. Define the function $\psi(y_1, y_2)$ by formula (3.3). It results from (3.4) that

$$\varphi(n_1 y_1 + \dots + n_k y_k) = \langle An, n \rangle$$

with $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$, $A = (\alpha_{ij})_{i,j=1}^k$, and $\alpha_{ij} = \psi(y_i, y_j)$. Since $\varphi(y) \geq 0$, the matrix A is positive semidefinite. Hence the function $\exp\{-\varphi(n_1 y_1 + \dots + n_k y_k)\}$ considered as a function of integer-valued variables n_j, is a positive definite function on the group $\mathbb{Z}^k$. It follows from what has been said that $f(y) = \exp\{-\varphi(y)\}$ is a positive definite function. Hence by the Bochner theorem, $f(y)$ is a characteristic function.

Remark 3.5. If $\varphi(y)$ is a continuous function on a group Y satisfying equation 2.16 (ii), then $\varphi(y + y_0) = \varphi(y)$ for all $y \in Y$, $y_0 \in b_Y$, in particular, $\varphi(y) = 0$ for all $y \in b_Y$. Indeed, let H be a compact subgroup of Y. Consider equation 2.16 (ii), assuming that $u, v \in H$. Integrating equation 2.16 (ii) over the group H with respect to the Haar distribution $dm_H(u)$ and taking into account 2.14 (a), we infer that $\varphi(v) = 0$ for all $v \in H$. Hence $\varphi(v) = 0$ when $v \in b_Y$. Fix $y \in Y$ and $y_0 \in b_Y$. Denote by M the smallest closed subgroup of Y containing element y_0. Since $y_0 \in b_Y$, the subgroup M is compact. Consider the function $P(l) = \varphi(y + l y_0)$ on the group $\mathbb{Z}$. It is easy to see that the function $P(l)$ satisfies the equation

$$P(m+n) + P(m-n) = 2P(m), \quad m, n \in \mathbb{Z}.$$

This implies that $P(l) = a_0 + l a_1, l \in \mathbb{Z}$, where a_0, a_1 are some constants depending, generally speaking, on y and y_0. The function $\varphi(y)$ is continuous on the compact set $y + M$, and hence it is bounded on $y + M$. Therefore $a_1 = 0$. It follows from this that $\varphi(y + y_0) = \varphi(y)$.

Let us study the support of a Gaussian distribution. Without loss of generality, we can assume that the Gaussian distribution is symmetric.

Proposition 3.6. *If $\gamma \in \Gamma^s(X)$, then $\sigma(\gamma) = G$, where G is a connected subgroup of the group X.*

Proof. Consider the set $E = \{y \in Y : \hat{\gamma}(y) = 1\}$, and set $G = A(X, E)$. It results from Proposition 2.13 that $\sigma(\gamma) \subset G$. Hence we may consider γ as a Gaussian distribution on the group G. Set $H = G^*$. The characteristic function $\hat{\gamma}(y)$, $y \in H$ has the property

$$\{h \in H : \hat{\gamma}(h) = 1\} = \{0\}.$$

By Remark 3.5, $b_H = \{0\}$. This implies by Theorem 1.9.3 that $c_G = G$, i.e., G is a connected group.

Proposition 3.6 will be proved if we verify that $\gamma(U) > 0$ for any open subset $U \subset G$. Suppose that $\gamma(U) = 0$ for some open subset $U \subset G$. Choose an open subset U_0 in U and a neighbourhood of zero V in G such that $U_0 + V \subset U$. By Theorem 1.11.2, $G = L \times K$, where $L \cong \mathbb{R}^m$, $m \geq 0$, and K is a compact connected group. Apply Theorem 1.12.2 and choose in $K \cap V$ a closed subgroup S such that $K/S \cong \mathbb{T}^k \times F$, where F is a finite group. Since the group K is connected, $F = \{0\}$. It follows from this that the factor group G/S is topologically isomorphic to the group $\mathbb{R}^m \times \mathbb{T}^k$. Denote by τ this topological isomorphism, and set $p = \tau \circ \pi$, where π is the natural homomorphism $\pi\colon G \mapsto G/S$. It is obvious that $p(\gamma) \in \Gamma(\mathbb{R}^m \times \mathbb{T}^k)$, and

$$\{y \in \mathbb{R}^m \times \mathbb{Z}^k : \widehat{p(\gamma)}(y) = 1\} = \{0\}.$$

Moreover

$$p(\gamma)(p(U_0)) = \gamma\{p^{-1}(p(U_0))\} = \gamma\{U_0 + S\} \leq \gamma\{U_0 + V\} \leq \gamma\{U\} = 0,$$

but this is impossible because $p(U_0)$ is an open set. The contradiction obtained proves that $\gamma(U) > 0$. □

The proof of Proposition 3.6 implies directly the following statement:

Corollary 3.7. *Let γ be a symmetric Gaussian distribution on a group X. Assume that $\{y \in Y : \hat{\gamma}(y) = 1\} = \{0\}$. This implies that the group X is connected, and $\sigma(\gamma) = X$.*

Thus studying Gaussian distributions on a group X we can restrict ourselves to the case when X is a connected group. We will prove now that an arbitrary symmetric Gaussian distribution on a connected group X is a continuous homomorphic image of a Gaussian distribution in a linear space. This space is either finite-dimensional or infinite-dimensional depending on the dimension of the group X. First consider the case when the group X has finite dimension.

Proposition 3.8. *Let X be a connected group of finite dimension l. Then there exists a continuous homomorphism $p\colon \mathbb{R}^l \mapsto X$ with the property: for any symmetric Gaussian distribution $\gamma \in \Gamma^s(X)$ there is a symmetric Gaussian distribution $\mu \in \Gamma^s(\mathbb{R}^l)$ such that $\gamma = p(\mu)$.*

Proof. By Theorem 1.11.2 the group X is topologically isomorphic to the group $\mathbb{R}^m \times K$, where $m \geq 0$, and K is a compact connected group. First assume that $X = K$. Put $D = K^*$. We conclude from Theorems 1.6.1 and 1.6.2 that D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = l$. Choose in D a maximal independent system of elements $d_1, \ldots, d_l$. Then for every $d \in D$ there exist integers $k, k_1, \ldots, k_l$ such that

$$kd = k_1 d_1 + \cdots + k_l d_l. \tag{3.5}$$

The independence of the set $\{d_1, \ldots, d_l\}$ implies that the rational numbers $\{k_j/k\}_{j=1}^l$ are uniquely determined by d. Since D is a torsion-free group, the mapping

$$\pi d = (k_1/k, \ldots, k_l/k) \tag{3.6}$$

is a monomorphism $\pi\colon D \mapsto \mathbb{R}^l$.

Let a function $\varphi(d)$ on the group D satisfy equation 2.16 (ii). Taking into account Remark 3.3 we define the function $\psi(y_1, y_2)$ by formula (3.3). It results from (3.5) and (3.4) that

$$k^2\varphi(d) = k^2\psi(d, d) = \psi(kd, kd) = \psi(k_1d_1 + \cdots + k_ld_l, k_1d_1 + \cdots + k_ld_l)$$
$$= \sum_{i,j=1}^{l} \alpha_{ij} k_i k_j,$$

where $\alpha_{ij} = \psi(d_i, d_j)$, $1 \le i, j \le l$. Hence

$$\varphi(d) = \sum_{i,j=1}^{l} \alpha_{ij}(k_i/k)(k_j/k) = \langle A\pi d, \pi d\rangle, \tag{3.7}$$

where $A = (\alpha_{ij})_{i,j=1}^{l}$. If $\varphi(d) \ge 0$ for all $d \in D$, then the symmetric matrix A is positive semidefinite, i.e., the square form $\langle As, s\rangle$ is nonnegative for all $s \in \mathbb{R}^l$.

Let $\gamma \in \Gamma^s(K)$. Assume that the function $\varphi(d)$ in 3.1 (i) corresponds to the characteristic function $\hat{\gamma}(d)$. We deduce from (3.7) that

$$\hat{\gamma}(d) = \exp\{-\langle A\pi d, \pi d\rangle\}, \quad d \in D. \tag{3.8}$$

Let μ be a Gaussian distribution on the group $\mathbb{R}^l$ with the characteristic function

$$\hat{\mu}(s) = \exp\{-\langle As, s\rangle\}, \quad s \in \mathbb{R}^l. \tag{3.9}$$

Denote by $p\colon \mathbb{R}^l \mapsto K$ the homomorphism adjoint to the homomorphism π. It follows from Proposition 2.10 and (3.9) that

$$\widehat{p(\mu)}(d) = \exp\{-\langle A\pi d, \pi d\rangle\}, \quad d \in D. \tag{3.10}$$

From (3.8), (3.10) and 2.7 (b) we conclude that $\gamma = p(\mu)$.

Assume now that the group X is noncompact. Then $X \cong \mathbb{R}^m \times K$, where $m \ge 1$, and K is a compact connected group of finite dimension n, $m + n = l$. Put $D = K^*$.

Then by Theorem 1.7.1, $Y \cong \mathbb{R}^m \times D$. By Theorems 1.6.1 and 1.6.2, D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = n$.

To avoid introducing new notation we will suppose that $Y = \mathbb{R}^m \times D$. Denote by H the subgroup of Y of the form $H = \mathbb{Q}^m \times D$. It is obvious that H is dense in Y. Consider in $\mathbb{Q}^m$ a maximal independent system of elements $e_1 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)$. Let $d_1, \dots, d_n$ be a maximal independent system of elements in D. Then $e_1, \dots, e_m, d_1, \dots, d_n$ are a maximal independent system of elements in H. Denote by $y = (s, d)$, $s \in \mathbb{R}^m$, $d \in D$ elements of the group Y. Let $h \in H$. Then there exist integers $k, p_1, \dots, p_m, k_1, \dots, k_n$ such that

$$kh = p_1e_1 + \cdots + p_me_m + k_1d_1 + \cdots + k_nd_n.$$

The mapping

$$\pi h = (p_1/k, \dots, p_m/k, k_1/k, \dots, k_n/k)$$

is a monomorphism $\pi: H \mapsto \mathbb{R}^l$. As is easily seen, the restriction of π to $\mathbb{Q}^m$ is the identity mapping. Hence the monomorphism π can be extended by continuity from H to Y by the formula

$$\pi(s,d) = (s, k_1/k, \dots, k_n/k), \quad (s,d) \in Y \tag{3.11}$$

(we keep the notation π for the extended monomorphism). Reasoning as in the case when $X = K$, we make sure that the restriction of the function $\varphi(y)$ to H has the form

$$\varphi(h) = \langle A\pi h, \pi h\rangle, \quad h \in H, \tag{3.12}$$

where $A = (\alpha_{ij})_{i,j=1}^{l}$ is a symmetric positive semidefinite matrix, $\alpha_{ij} = \psi(e_i, e_j)$, $1 \le i, j \le m$, $\alpha_{i,j+m} = \psi(e_i, d_j)$, $1 \le i \le m$, $1 \le j \le n$, $\alpha_{i+m,j+m} = \psi(d_i, d_j)$, $1 \le i, j \le n$. Since the function $\varphi(y)$ is continuous on Y, equality (3.12) is true for every $y \in Y$. Furthermore π in the right side of (3.12) is defined by (3.11). The proof can be completed as in the case where $X = K$. □

Corollary 3.9. *Let X be a connected group of finite dimension l. Then any symmetric Gaussian distribution γ on X is a continuous monomorphic image of a Gaussian distribution on a group $\mathbb{R}^m \times \mathbb{T}^n$.*

Proof. Let $\mu \in \Gamma^s(\mathbb{R}^l)$ and $p: \mathbb{R}^l \mapsto X$ be as in Proposition 3.8. We note that $L = \sigma(\mu)$ is a subspace in $\mathbb{R}^l$. Denote by q the restriction of p to L. Put $E = \operatorname{Ker} q$. Then by Theorem 1.16, $E \cong \mathbb{R}^a \times \mathbb{Z}^n$. Denote by τ the natural homomorphism $\tau: L \mapsto L/E$. It is obvious that $L/E \cong \mathbb{R}^m \times \mathbb{T}^n$. To avoid introducing new notation we will suppose that $L/E = \mathbb{R}^m \times \mathbb{T}^n$. Then $\tau(\mu) \in \Gamma(\mathbb{R}^m \times \mathbb{T}^n)$. Set $r = q\tau^{-1}$. It is obvious that r is a continuous monomorphism from $\mathbb{R}^m \times \mathbb{T}^n$ to X and $\gamma = r(\tau(\mu))$. □

Consider now the case when a connected group X has infinite dimension. We recall that the group X is assumed to be second countable. This implies by Theorems 1.6.3 and 1.11.2 that if the dimension of the group X is infinite, then $\dim X = \aleph_0$. In what follows we need the concept of a Gaussian distribution in the space $\mathbb{R}^{\aleph_0}$.

3.10 Gaussian distributions in the space $\mathbb{R}^{\aleph_0}$. Denote by $\mathbb{R}^{\aleph_0}$ the space of all sequences of real numbers in the product topology. The convergence of elements

$$t^{(k)} = (t_1^{(k)}, \dots, t_n^{(k)}, \dots) \to t = (t_1, \dots, t_n, \dots)$$

in this topology is the coordinate-wise convergence. The space $\mathbb{R}^{\aleph_0}$ can be regarded as the projective limit of a directed set of spaces $\mathbb{R}^n$. We note that the topological group $\mathbb{R}^{\aleph_0}$ is not locally compact. It is possible to introduce a metric in the group $\mathbb{R}^{\aleph_0}$ inducing the product topology by the formula

$$\rho(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{1 + |u_n - v_n|},$$

$u = (u_1, \dots, u_n, \dots), v = (v_1, \dots, v_n, \dots) \in \mathbb{R}^{\aleph_0}$. With respect to this metric $\mathbb{R}^{\aleph_0}$ is a complete separable group.

Denote by $\mathbb{R}^{\aleph_0 *}$ the space of all finitary sequences of real numbers with the topology of strictly inductive limit of spaces $\mathbb{R}^n$. The convergence of elements

$$s^{(k)} = (s_1^{(k)}, \dots, s_n^{(k)}, 0, \dots) \to s = (s_1, \dots, s_n, 0, \dots)$$

in this topology means that all $s^{(k)}$ belong to one space $\mathbb{R}^n$ and converge there. The topological group $\mathbb{R}^{\aleph_0 *}$ is not locally compact either. Let $t = (t_1, \dots, t_n, \dots) \in \mathbb{R}^{\aleph_0}$ and $s = (s_1, \dots, s_n, 0, \dots) \in \mathbb{R}^{\aleph_0 *}$. Set $\langle t, s \rangle = \sum_{j=1}^{\infty} t_j s_j$. Fix $s \in \mathbb{R}^{\aleph_0 *}$ and consider the function

$$(t, s) = \exp\{i \langle t, s \rangle\} \tag{3.13}$$

on the group $\mathbb{R}^{\aleph_0}$. This function is a continuous homomorphism of the group $\mathbb{R}^{\aleph_0}$ into the circle group $\mathbb{T}$, i.e., it is a character of the group $\mathbb{R}^{\aleph_0}$. It is easily seen that every character of the group $\mathbb{R}^{\aleph_0}$ is of the form (3.13). Denote by $\mathfrak{B}(\mathbb{R}^{\aleph_0})$ the σ-algebra of Borel sets in $\mathbb{R}^{\aleph_0}$, i.e., the smallest σ-algebra generated by the sets of the form $U_J^n \times \mathbb{R}^{\aleph_0 \setminus J}$, where U_J^n is an open set in $\mathbb{R}_J^n$, $J = (j_1, \dots, j_n)$, $\mathbb{R}_J^n = \{t \in \mathbb{R}^{\aleph_0} : t_j = 0 \text{ for } j \notin J\}$, and $\mathbb{R}^{\aleph_0 \setminus J} = \{t \in \mathbb{R}^{\aleph_0} : t_j = 0 \text{ for all } j \in J\}$. Taking into account the definition of the topology in $\mathbb{R}^{\aleph_0}$ and the σ-algebra $\mathfrak{B}(\mathbb{R}^{\aleph_0})$, it is easy to see that every continuous function on the group $\mathbb{R}^{\aleph_0}$ is $\mathfrak{B}(\mathbb{R}^{\aleph_0})$-measurable. Let μ be a distribution on $\mathbb{R}^{\aleph_0}$. We define the characteristic function of μ by the formula

$$\hat{\mu}(s) = \int\limits_{\mathbb{R}^{\aleph_0}} (t, s)\, d\mu(t), \quad s \in \mathbb{R}^{\aleph_0 *}.$$

It is easily seen that the characteristic function $\hat{\mu}(s)$ has the properties:

(a) $\hat{\mu}(0) = 1$;

(b) the function $\hat{\mu}(s)$ is continuous;

(c) the function $\hat{\mu}(s)$ is positive definite on every subspace $\mathbb{R}_J^n \subset \mathbb{R}^{\aleph_0 *}$.

It results from Kolmogorov's theorem and from Bochner's theorem for the group $\mathbb{R}^n$ that every function $g(s)$ on the group $\mathbb{R}^{\aleph_0 *}$ satisfying properties (a)–(c) defines a unique distribution μ_g on the group $\mathbb{R}^{\aleph_0}$ such that $\hat{\mu}_g(s) = g(s)$. Let $A = (\alpha_{ij})_{i,j=1}^{\infty}$ be a symmetric positive semidefinite matrix, i.e., the square form $\langle As, s \rangle = \sum_{i,j=1}^{\infty} \alpha_{ij} s_i s_j$ is nonnegative for all $s \in \mathbb{R}^{\aleph_0 *}$. We can define now a Gaussian distribution on the group $\mathbb{R}^{\aleph_0}$.

A distribution μ on the group $\mathbb{R}^{\aleph_0}$ is called *Gaussian* if its characteristic function is represented in the form

$$\hat{\mu}(s) = (t, s) \exp\{-\langle As, s \rangle\}, \quad s \in \mathbb{R}^{\aleph_0 *}, \tag{3.14}$$

where $t \in \mathbb{R}^{\aleph_0}$, and $A = (\alpha_{ij})_{i,j=1}^{\infty}$ is a symmetric positive semidefinite matrix.

Denote by $\Gamma(\mathbb{R}^{\aleph_0})$ the set of Gaussian distributions on the group $\mathbb{R}^{\aleph_0}$. A Gaussian distribution is called *symmetric* if in (3.14) $t = 0$. Denote by $\Gamma^s(\mathbb{R}^{\aleph_0})$ the set of symmetric Gaussian distributions on the group $\mathbb{R}^{\aleph_0}$.

We will prove now an analogue of Proposition 3.8 for connected groups of infinite dimension.

Proposition 3.11. *Let X be a connected group and let* $\dim X = \aleph_0$. *Then there exists a continuous homomorphism $p\colon \mathbb{R}^{\aleph_0} \mapsto X$ with the property: for any symmetric Gaussian distribution $\gamma \in \Gamma^s(X)$ there is a symmetric Gaussian distribution $\mu \in \Gamma^s(\mathbb{R}^{\aleph_0})$ such that $\gamma = p(\mu)$.*

Proof. By Theorem 1.11.2 the group X is topologically isomorphic to the group $\mathbb{R}^m \times K$, where $m \geq 0$, and K is a compact connected group. First assume that $X = K$. Put $D = K^*$. Then by Theorems 1.6.1 and 1.6.2, D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = \aleph_0$. Choose in D a maximal independent system of elements $d_1, \dots, d_l, \dots$. Then for every $d \in D$ there exist integers $k, k_1, \dots, k_l$ such that

$$kd = k_1 d_1 + \cdots + k_l d_l.$$

The independence of the set $\{d_1, \dots, d_l\}$ implies that the rational numbers $\{k_j/k\}_{j=1}^{l}$ are uniquely determined by d. Since D is a torsion-free group, the mapping

$$\pi d = (k_1/k, \dots, k_l/k, 0, \dots) \tag{3.15}$$

is a monomorphism $\pi : D \mapsto \mathbb{R}^{\aleph_0 *}$.

Define the mapping $p\colon \mathbb{R}^{\aleph_0} \mapsto K$ by the equality $(pt, d) = (t, \pi d)$ for all $t \in \mathbb{R}^{\aleph_0}$, $d \in D$. It is easy to verify that p is a continuous homomorphism. We observe that if α is an arbitrary distribution on the group $\mathbb{R}^{\aleph_0}$, then the characteristic function of the distribution $p(\alpha) \in \mathrm{M}^1(K)$ is of the form

$$\widehat{p(\alpha)}(d) = \hat{\alpha}(\pi d), \quad d \in D. \tag{3.16}$$

Note that (3.16) does not follow from Proposition 2.10 but must be proved independently, because the groups $\mathbb{R}^{\aleph_0}$ and $\mathbb{R}^{\aleph_0 *}$ are not locally compact.

Just as in the proof of Proposition 3.8 it is easy to make sure that every nonnegative function $\varphi(d)$ on the group D satisfying equation 2.16 (ii) is represented in the form

$$\varphi(d) = \langle A\pi d, \pi d\rangle, \quad d \in D, \tag{3.17}$$

where $A = (\alpha_{ij})_{i,j=1}^{\infty}$ is a symmetric positive semidefinite matrix, $\alpha_{ij} = \psi(d_i, d_j)$, and πd is defined by (3.15).

Let $\gamma \in \Gamma^s(K)$ and let the function $\varphi(d)$ correspond to the characteristic function $\hat{\gamma}(d)$ in 3.1 (i). We deduce from (3.17) that

$$\hat{\gamma}(d) = \exp\{-\langle A\pi d, \pi d\rangle\}, \quad d \in D. \tag{3.18}$$

Let μ be a Gaussian distribution on the group $\mathbb{R}^{\aleph_0}$ with the characteristic function

$$\hat{\mu}(s) = \exp\{-\langle As, s\rangle\}, \quad s \in \mathbb{R}^{\aleph_0 *}. \tag{3.19}$$

It follows from (3.16) and (3.19) that

$$\widehat{p(\mu)}(d) = \exp\{-\langle A\pi d, \pi d\rangle\}, \quad d \in D. \tag{3.20}$$

Comparing (3.18) and (3.20) and using 2.7 (b), we conclude that $\gamma = p(\mu)$.

The general case, i.e., when $X \cong \mathbb{R}^m \times K$, where $m \geq 1$, and K is a compact connected group, dim $K = \aleph_0$, can be considered as in Proposition 3.8. In so doing the monomorphism $\pi : Y \mapsto \mathbb{R}^{\aleph_0 *}$ is defined by

$$\pi(s, d) = (s, k_1/k, \ldots, k_l/k, 0, \ldots), \quad (s, d) \in Y. \tag{3.21}$$

□

Remark 3.12. It was proved in Proposition 3.6 that the support of a symmetric Gaussian distribution on a group X is a connected subgroup G of X. It results from Propositions 3.8 and 3.11 that for every connected group X there exists a symmetric Gaussian distribution $\gamma \in \Gamma^s(X)$ such that $\sigma(\gamma) = X$. Indeed, since on the group $\mathbb{R}^m$ there exists a symmetric Gaussian distribution $\gamma \in \Gamma^s(\mathbb{R}^m)$ such that $\sigma(\gamma) = \mathbb{R}^m$, by Theorem 1.11.2 it suffices to prove this statement for a compact connected group. So, let us assume that a group X is connected and compact, and π is either a homomorphism such as in the proof of Proposition 3.8 if dim $X = l < \infty$, or such as in the proof of Proposition 3.11 if dim $X = \aleph_0$. In both cases the required distribution $\gamma \in \Gamma(X)$ is a distribution γ with the characteristic function

$$\hat{\gamma}(y) = \exp\{-\langle \pi y, \pi y\rangle\}, \quad y \in D.$$

This follows from the fact that $\hat{\gamma}(y) = 1$ if and only if $y = 0$ and from Corollary 3.7.

It should be noted that the proofs of Propositions 3.8 and 3.11 imply directly that if $f(y) = \exp\{-\varphi(y)\}$, where $\varphi(y)$ is a continuous nonnegative function on the group Y satisfying equation 2.16 (ii), then $f(y)$ is a characteristic function (compare with Remark 3.4).

3.13. Let X be a nondiscrete group and γ be a measure on X. Expand the measure γ into the sum

$$\text{(i)} \quad \gamma = \gamma_{ac} + \gamma_s + \gamma_d,$$

where γ_{ac} is a measure absolutely continuous with respect to m_X, γ_s is a measure singular with respect to m_X, and γ_d is a discrete measure. We will call such expansion the *structure* of the measure γ. Let us study the structure of a Gaussian distribution. Taking into account Proposition 3.6 we can assume that the group X is connected. We first verify that if γ is a nondegenerate Gaussian distribution on X, then $\gamma_d = 0$ in (i). By Theorem 1.11.2, $X \cong \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact connected group. Put $D = K^*$. By Theorems 1.7.1, 1.6.1, and 1.6.2, $Y \cong \mathbb{R}^m \times D$, where D is a discrete torsion-free group. Let $y_0 \in Y$ be an arbitrary element. Denote by M the subgroup of Y generated by the element y_0. It is obvious that $M \cong \mathbb{Z}$, and hence $M^* \cong \mathbb{T}$. By Theorem 1.9.2, $M^* \cong X/A(X, M)$. Thus $X/A(X, M) \cong \mathbb{T}$. Denote this topological isomorphism by τ and set $p = \tau \circ \pi$, where $\pi\colon X \mapsto X/A(X, M)$ is the natural homomorphism. Then $p\colon X \mapsto \mathbb{T}$ is a continuous homomorphism. If $\gamma(\{x_0\}) > 0$, then we have $p(\gamma)(\{px_0\}) = \gamma(p^{-1}(\{px_0\})) \geq \gamma(\{x_0\}) > 0$, but this is impossible because $p(\gamma) \in \Gamma(\mathbb{T})$, and every nondegenerate Gaussian distribution on the circle group $\mathbb{T}$ is absolutely continuous.

We will study the structure of Gaussian distributions separately for not locally connected groups and for locally connected groups. We recall that by Theorem 1.11.3 every second countable connected locally connected locally compact Abelian group X is topologically isomorphic to the group $\mathbb{R}^m \times \mathbb{T}^{\mathfrak{n}}$, where $m \geq 0$ and $\mathfrak{n} \leq \aleph_0$.

Proposition 3.14. *Let X be a connected and not locally connected group, and γ be a nondegenerate symmetric Gaussian distribution on X. Then in 3.13* (ii)*, $\gamma = \gamma_s$.*

Proof. Denote by L either the group $\mathbb{R}^m$, where $m \geq 0$ if $\dim X = m$, or the group $\mathbb{R}^{\aleph_0}$ if $\dim X = \aleph_0$. It follows from Propositions 3.8 and 3.11 that there exist a continuous homomorphism $p\colon L \mapsto X$ and a Gaussian distribution ν on L such that $\gamma = p(\nu)$. Hence the distribution γ is concentrated on the subgroup $p(L)$. Let us verify that $p(L)$ is a Borel subgroup. Put $G = L/\operatorname{Ker} p$. The homomorphism p defines the continuous monomorphism $\pi\colon G \mapsto X$ by the formula $\pi[t] = pt$. We note that because L is a complete separable metric group and the kernel $\operatorname{Ker} p$ is a closed subgroup of L, then G is also a complete separable metric group. Applying the Suslin theorem to the mapping π and taking into account that $\pi(G) = p(L)$, we conclude that $p(L)$ is a Borel set. The subgroup $p(L)$ can not coincide with the group X. Indeed, if $X = p(L)$, then the group X is the union of its one-parameter subgroups, and hence X must be arcwise connected. Then by Theorem 1.11.3, $X \cong \mathbb{R}^m \times \mathbb{T}^{\mathfrak{n}}$, which contradicts the condition of the proposition.

Now we note that if a group X is second countable and B is a Borel set in X such that $m_X(B) > 0$, then the difference set $B - B = \{x \in X : x = u - v,\ u, v \in B\}$ contains a neighbourhood of zero of the group X ([59], (20.17)). This implies that if X is a connected group and B is a measurable subgroup of X such that $m_X(B) > 0$, then $B = X$. It follows from what has been said above that $m_X(p(L)) = 0$. Hence $\gamma = \gamma_s$. □

3.15. Assume now that a group X is connected and locally connected. If $\dim X = l < \infty$, then by Theorem 1.11.3, $X \cong \mathbb{R}^m \times \mathbb{T}^n$, where $m \geq 0$, $n \geq 0$, $m + n = l$. Let γ be a symmetric nondegenerate Gaussian distribution on X. By Proposition 3.8, $\gamma = p(\mu)$, where μ is a Gaussian distribution in $\mathbb{R}^l$. Set $G = \sigma(\mu)$. Then G is a subspace in $\mathbb{R}^l$. It follows from this that if $G = \mathbb{R}^l$, then $\gamma = \gamma_{ac}$, and if $G \neq \mathbb{R}^l$, then $\gamma = \gamma_s$.

Assume now that $\dim X = \aleph_0$. Then by Theorem 1.11.3, $X \cong \mathbb{R}^m \times \mathbb{T}^{\aleph_0}$, where $m \geq 0$. Let us now make use of the following theorem by Kakutani ([67]).

Theorem. *Let $\{\mu_i\}$ and $\{\nu_i\}$ be two sequences of distributions on a probability space Ω. Assume that for all $i = 1, 2, \ldots$ the distributions μ_i and ν_i are mutually absolutely* *continuous. Then the direct products of the distributions*

$$\text{(i)} \quad \mu = \bigotimes_{i=1}^{\infty} \mu_i, \quad \nu = \bigotimes_{i=1}^{\infty} \nu_i$$

are mutually absolutely continuous, if the infinite product

$$\text{(ii)} \quad \prod_{i=1}^{\infty} \int_{\Omega} \sqrt{\frac{d\mu_i}{d\nu_i}}\, d\nu_i$$

converges ($d\mu_i/d\nu_i$ *is the Radon–Nikodym derivative*)*, they are mutually singular if this product diverges.*

The Kakutani theorem allows us to construct on the group X both absolutely continuous Gaussian distributions and singular Gaussian distributions. To simplify notation we suppose that $X = \mathbb{T}^{\aleph_0}$. Then by Proposition 3.11 the characteristic function of a distribution $\mu \in \Gamma^s(\mathbb{T}^{\aleph_0})$ is of the form

$$\hat{\mu}(n) = \exp\{-\langle An, n\rangle\}, \quad n = (n_1, n_2, \dots) \in \mathbb{Z}^{\aleph_0 *}, \tag{3.22}$$

where $A = (\alpha_{ij})_{i,j=1}^{\infty}$ is a symmetric positive semidefinite matrix.

Let μ_i be a symmetric Gaussian distribution on the circle group $\mathbb{T}$ with the characteristic function

$$\hat{\mu}_i(n) = \exp\{-\sigma_i^2 n^2/2\}, \quad \sigma_i > 0,\ n \in \mathbb{Z},$$

and let $\nu_i = m_{\mathbb{T}}$. Then in (i) $\mu \in \Gamma(\mathbb{T}^{\aleph_0})$, and the matrix $A = (\alpha_{ij})_{i,j=1}^{\infty}$ corresponding to the Gaussian distribution μ in representation (3.22) is diagonal, i.e., $\alpha_{ij} = 0$ for all $i \neq j$, $\alpha_{ii} = \frac{1}{2}\sigma_i^2$. Moreover $\nu = m_{\mathbb{T}^{\aleph_0}}$. We note that the density $\frac{d\mu_i}{d\nu_i}(e^{it})$ can be written in the form

$$\frac{d\mu_i}{d\nu_i}(e^{it}) = \sum_{n=-\infty}^{\infty} \exp\{-\sigma_i^2 n^2/2 + int\}.$$

It is easily seen that if $\sigma_i^2 = i$, $i = 1, 2, \dots$, then infinite product (ii) converges and hence $\mu = \mu_{ac}$. If $\sigma_i^2 = 1$, $i = 1, 2, \dots$, then infinite product (ii) diverges and hence $\mu = \mu_s$.

An interesting and unsolved problem is to find out whether there exists a Gaussian distribution γ on the infinite-dimensional torus $\mathbb{T}^{\aleph_0}$ such that in 3.13 (i) $\gamma_{ac} > 0$ and $\gamma_s > 0$.

Now we will formulate two criteria in order that a distribution γ on the group X be Gaussian.

Proposition 3.16. *A distribution γ on a group X is Gaussian if and only if the following conditions are satisfied:*

(i) *γ is an infinitely divisible distribution;*

(ii) *if* $\gamma = e(\Phi) * \alpha$, *where* Φ *is a finite measure on* X, *and* α *is an infinitely divisible distribution, then the measure* Φ *is degenerated at zero.*

The proof follows directly from the Lévy–Khinchin formula and from the uniqueness of the function $\varphi(y)$ in representation 2.16 (i).

Proposition 3.17. *A distribution* γ *on a group* X *is Gaussian if and only if the following conditions are satisfied:*

(i) γ *is an infinitely divisible distribution;*

(ii) $y(\gamma) \in \Gamma(\mathbb{T})$ *for all characters* $y \in Y$.

Proof. It is obvious that a Gaussian distribution on a group X satisfies conditions (i) and (ii). Let γ be an infinitely divisible distribution satisfying condition (ii). Assume that $\gamma = e(\Phi) * \alpha$, where Φ is a finite measure on X, and α is an infinitely divisible distribution. Then $y(\gamma)$ and $y(\alpha)$ are infinitely divisible distributions on the circle group $\mathbb{T}$, and $y(\gamma) = e(y(\Phi)) * y(\alpha)$. By condition (ii), $y(\gamma) \in \Gamma(\mathbb{T})$. By Proposition 3.16 the measure $y(\Phi)$ is degenerated at zero. Hence the measure Φ is also degenerated at zero. Applying Proposition 3.16 once again we obtain that $\gamma \in \Gamma(X)$. □

In what follows we need the following statement.

Lemma 3.18. *Let* H *be an open subgroup of a group* Y, $\varphi_0(h)$ *be a continuous nonnegative function on* H *satisfying equation* 2.16 (ii). *Then there exists a continuous nonnegative function* $\varphi(y)$ *on* Y *satisfying equation* 2.16 (ii) *and such that its restriction to* H *coincides with* $\varphi_0(h)$.

Proof. Let $y_1 \notin H$. The standard reasoning show, that it suffices to extend the function $\varphi_0(h)$ retaining its properties from the subgroup H to the open subgroup $H_1 = \{y \in Y : y = ny_1 + h,\ n \in \mathbb{Z},\ h \in H\}$. Two cases are possible:

1. $ny_1 \cap H = \emptyset$ for all $n \in \mathbb{Z}, n \neq 0$. Set $\varphi(ny_1 + h) = \varphi_0(h), n \in \mathbb{Z}, h \in H$.

2. $ny_1 \in H$ for some $n \in \mathbb{Z}, n \neq 0$. Let n_0 be the minimal natural number such that $n_0 y_1 \in H$. Then $n_0 y \in H$ for all $y \in H_1$. Set $\varphi(y) = \varphi_0(n_0 y)/n_0^2$, $y \in H_1$.

It is obvious that the function $\varphi(y)$ defined above is continuous and nonnegative. We have,

$$\begin{aligned}\varphi(u+v) + \varphi(u-v) &= \varphi_0(n_0(u+v))/n_0^2 + \varphi_0(n_0(u-v))/n_0^2 \\ &= 2(\varphi_0(n_0 u)/n_0^2 + \varphi_0(n_0 v)/n_0^2) \\ &= 2[\varphi(u) + \varphi(v)].\end{aligned}$$

Thus the function $\varphi(y)$ has the desired properties. □

3.19. In conclusion we will prove some statements about the properties of the class $\Gamma(X)$.

(a) The set $\Gamma(X)$ is a subsemigroup of $\mathrm{M}^1(X)$.

The proof follows directly from Definition 3.1.

(b) Let K be a compact subgroup of the group X, $\gamma_0 \in \Gamma(X/K)$, and $p\colon X \mapsto X/K$ be the natural homomorphism. Then there exists a Gaussian distribution $\gamma \in \Gamma(X)$ such that $\gamma_0 = p(\gamma)$.

To prove this, set $H = A(Y, K)$. By Theorem 1.9.4, H is an open subgroup of Y. It results from Theorems 1.9.1 and 1.9.2 that $(X/K)^* \cong H$. For this reason the characteristic function $\hat{\gamma}_0(h)$ is of the form $\hat{\gamma}_0(h) = ([x_0], h)\exp\{-\varphi_0(h)\}$, where $[x_0] \in X/K$ and $\varphi_0(h)$ is a continuous nonnegative function on H satisfying equation 2.16 (ii). Take $x \in [x_0]$. By Lemma 3.18 there exists a function $\varphi(y)$ extending the function $\varphi_0(h)$ from the subgroup H to Y and retaining its properties. Consider on the group Y the function $f(y) = (x, y)\exp\{-\varphi(y)\}$. By Remark 3.4, $f(y)$ is the characteristic function of a Gaussian distribution γ on X. By Corollary 2.11, $\gamma_0 = p(\gamma)$.

(c) Denote by $I_C(X)$ the set of the Haar distributions of compact connected subgroups of the group X. Then the equality

$$\overline{\Gamma(X)} = \Gamma(X) * I_C(X) \tag{3.23}$$

holds true.

Let us first prove the inclusion

$$\overline{\Gamma(X)} \supset \Gamma(X) * I_C(X). \tag{3.24}$$

Let K be a compact connected subgroup of X, and $H = A(Y, K)$. By Remark 3.12 there is a symmetric Gaussian distribution $\gamma \in \Gamma^s(K)$ such that $\sigma(\gamma) = K$. The distribution γ can be considered as a Gaussian distribution on the group X. Let the characteristic function $\hat{\gamma}(y)$ have the representation

$$\hat{\gamma}(y) = \exp\{-\varphi(y)\}, \quad y \in Y.$$

It is obvious that $\varphi(y) = 0$ for all $y \in H$. By Proposition 2.13 it follows from $\sigma(\gamma) = K$ that $\varphi(y) > 0$ for all $y \notin H$. Let k be a natural number, and γ_k be a Gaussian distribution on X with the characteristic function

$$\hat{\gamma}_k(y) = \exp\{-k\varphi(y)\}, \quad y \in Y.$$

It is clear that $\hat{\gamma}_k(y) = 1$ for all $y \in H$, and $\lim_{k\to\infty} \gamma_k(y) = 0$ for all $y \notin H$. Taking into account 2.14 (i) we see that the sequence $\hat{\gamma}_k(y)$ pointwise converges to $\widehat{m}_K(y)$. It is easily seen that the sequence $\hat{\gamma}_k(y)$ converges to $\widehat{m}_K(y)$ uniformly on every compact subset of Y. We deduce from 2.7 (h) that $\gamma_k \Rightarrow m_K$, i.e., $m_K \in \overline{\Gamma(X)}$. This implies that for every distribution $\mu \in \Gamma(X)$ the convergence

$$\mu * \gamma_k \Rightarrow \mu * m_K$$

holds. Thus (3.24) is true.

Let us prove the converse inclusion. Let $\mu_k \in \Gamma(X)$, $\mu \in \mathrm{M}^1(X)$, and $\mu_k \Rightarrow \mu$. It follows from 2.7 (h) that $\hat{\mu}_k(y) \to \hat{\mu}(y)$ uniformly on every compact subset of Y. By 2.15 (c), μ is an infinitely divisible distribution. Therefore, by 2.15 (b) the set

$N = \{y \in Y : \hat{\mu}(y) \neq 0\}$ is an open subgroup of Y. By Theorem 1.9.4, $K = A(X, N)$ is a compact subgroup of X. Let $\hat{\mu}_k(y) = (x_k, y)\exp\{-\varphi_k(y)\}$, where $x_k \in X$, and $\varphi_k(y)$ is a continuous nonnegative function on Y satisfying equation 2.16 (ii). It is obvious that for all $y \in N$ there exists the limit

$$\lim_{k\to\infty} \varphi_k(y) = \varphi_0(y),$$

where $\varphi_0(y)$ is a continuous nonnegative function on Y satisfying equation 2.16 (ii). Since $(x_k, y) = \hat{\mu}_k(y)/|\hat{\mu}_k(y)|$, there exists the limit $\lim_{k\to\infty}(x_k, y)$ for all $y \in N$. Obviously, this limit is a character of the group N, and hence by Theorem 1.9.2 can be written in the form (x, y), where $x \in X$. By Lemma 3.18 there exists a function $\varphi(y)$, extending the function $\varphi_0(h)$ from the subgroup N to Y and retaining its properties. Consider on the group Y the function $f(y) = (x, y)\exp\{-\varphi(y)\}$. By Remark 3.4 $f(y)$ is the characteristic function of a Gaussian distribution γ on X. By 2.14 (i) and 2.7 (b), $\mu = \gamma * m_K$.

Let us verify that K is a connected group. By Theorem 1.6.2, it suffices to show that K^* is a torsion-free group. By Theorems 1.9.1 and 1.9.2, $K^* \cong Y/N$. Obviously, Y/N is a torsion-free group if and only if the subgroup N has the property: for every natural m if $my \in N$, then $y \in N$. We have

$$|\hat{\mu}(y)| = \lim_{k\to\infty} |\hat{\mu}_k(y)| = \lim_{k\to\infty} \exp\{-\varphi_k(y)\}.$$

This implies that $y \in N$ if and only if the sequence $\{\varphi_k(y)\}$ is bounded. Since $\varphi_k(my) = m^2\varphi_k(y)$, the boundedness of the sequence $\{\varphi_k(my)\}$ implies the boundedness of the sequence $\{\varphi_k(y)\}$. Thus if $my \in N$, then $y \in N$. We proved that

$$\overline{\Gamma(X)} \subset \Gamma(X) * I_C(X),$$

hence (3.23) is also proved.

4 Cramér's theorem on the decomposition of a Gaussian distribution on locally compact Abelian groups

Let X be a second countable locally compact Abelian group, Y be its character group. According to the classical Cramér theorem a Gaussian distribution on the real line has only Gaussian factors. On the other hand every nondegenerate Gaussian distribution on the circle group $\mathbb{T}$ has non-Gaussian factors. In this section we study the problem of decomposition of Gaussian distributions on the group X. We describe in particular all groups X on which every Gaussian distribution has only Gaussian factors.

4.1 Decomposition of a Gaussian distribution on the circle group $\mathbb{T}$. Let X be a compact group, $\gamma \in \mathrm{M}^1(X)$, and $0 < a < 1$. Assume that the distribution γ satisfies the condition

$$\text{(i)} \quad \gamma(E) \geq a m_X(E)$$

for any Borel set $E \in \mathfrak{B}(X)$. Then γ may be decomposed in the form

$$\text{(ii)} \quad \gamma = \frac{\gamma - am_X}{1-a} * [(1-a)E_0 + am_X].$$

To prove this it suffices to remove the parentheses on the right-hand side of (ii) and note that for every $\nu \in \mathrm{M}^1(X)$ the equality $\nu * m_X = m_X$ is valid.

If $\gamma \in \Gamma(\mathbb{T})$ is a nondegenerate distribution, then (3.2) implies that the distribution γ satisfies condition (i). Hence decomposition (ii) holds true. Obviously, both factors in (ii) are non-Gaussian. Thus any nondegenerate Gaussian distribution on the circle group $\mathbb{T}$ has non-Gaussian factors.

As appears from the above, the followign condition is necessary in order to assure that every Gaussian distribution on the group X has only Gaussian factors: the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. It turns out that this condition is also sufficient. To prove this statement we need the following lemmas.

Lemma 4.2. *Let X be a connected group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$, let* $\dim X = \aleph_0$, *and let* $\pi : Y \mapsto \mathbb{R}^{\aleph_0 *}$ *be the monomorphism defined by* (3.21). *Then the image $\pi(Y)$ is dense in $\mathbb{R}^{\aleph_0 *}$.*

Proof. By Theorem 1.11.2 the group X is topologically isomorphic to the group $\mathbb{R}^m \times K$, where $m \geq 0$, and K is a compact connected group, $\dim K = \aleph_0$. By Theorem 1.7.1, $Y \cong \mathbb{R}^m \times D$, where $D = K^*$. By Theorems 1.6.1 and 1.6.2, D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = \aleph_0$. To avoid introducing new notation we will assume that $Y = \mathbb{R}^m \times D$. Since the restriction of π to $\mathbb{R}^m$ coincides with the identity mapping, it suffices to prove the lemma in the case when $X = K$, $Y = D$. We will assume that for all n the group $\mathbb{Z}^n$ is embedded in the natural way in $\mathbb{R}^n$. Moreover the natural embeddings

$$\mathbb{R} \subset \mathbb{R}^2 \cdots \subset \mathbb{R}^n \subset \cdots \subset \mathbb{R}^{\aleph_0 *}$$

hold. Put $H_n = \overline{\pi(D)} \cap \mathbb{R}^n$. Then H_n is a closed subgroup of $\mathbb{R}^n$ and by Theorem 1.16.1, $H_n \cong \mathbb{R}^{l_n} \times \mathbb{Z}^{k_n}$. Since $\mathbb{Z}^n \subset H_n \subset \mathbb{R}^n$, we have $l_n + k_n = n$. We will verify that $k_n = 0$ for all n. Thus the lemma will be proved.

Assume the contrary, i.e., $k_n > 0$ for some n. We use the following property of closed subgroups of the group $\mathbb{R}^m$: a closed subgroup G of the group $\mathbb{R}^m$ is a direct product of its closed subgroup G_1 and another closed subgroup G_2, if and only if G_1 is an intersection of G with a subspace of $\mathbb{R}^m$ ([14], Chapter VII, § 1). It follows from $H_n = H_{n+1} \cap \mathbb{R}^n$ that H_n is a topological direct factor of H_{n+1}. Hence the group $\overline{\pi(D)}$ contains a subgroup F topologically isomorphic to $\mathbb{Z}$ as a direct factor, i.e., $\overline{\pi(D)} = F \times G$. Let e be a generator of the group F. Consider an arbitrary element $a_0 = e + g_0 \in (e + G) \cap \pi(D)$. Denote by M the subgroup of $\pi(D)$ generated by element a_0. Then obviously, $M \cong \mathbb{Z}$ and $\pi(D) = M \times (\pi(D) \cap G)$. Thus the group $\pi(D)$ contains a subgroup isomorphic to $\mathbb{Z}$ as a direct factor. Hence the group D contains a subgroup isomorphic to $\mathbb{Z}$ as a direct factor. We deduce from this that the group K contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$, contrary to the assumption of the lemma. □

Lemma 4.3. *Let X be a connected group of finite dimension l containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$, and let $\pi\colon Y \mapsto \mathbb{R}^l$ be the monomorphism defined by (3.11). Then the image $\pi(Y)$ is dense in $\mathbb{R}^l$.*

The proof of Lemma 4.3 is analogous to that of Lemma 4.2, and we omit it.

Lemma 4.4. *Let γ be a symmetric Gaussian distribution on a group X, and assume that it is a continuous monomorphic image of a Gaussian distribution either in the space $\mathbb{R}^l$ or in the space $\mathbb{R}^{\aleph_0}$. Then γ has only Gaussian factors.*

Proof. By Proposition 3.6 without loss of generality we can suppose that X is a connected group. Assume first that $\gamma = p(\mu)$, where $p\colon \mathbb{R}^l \mapsto X$ is a continuous monomorphism, $\mu \in \Gamma(\mathbb{R}^l)$. The distribution γ is concentrated on the subgroup $p(\mathbb{R}^l)$. Let $\gamma = \gamma_1 * \gamma_2$, $\gamma_j \in \mathrm{M}^1(X)$. By Proposition 2.2 the distributions γ_j can be replaced by their shifts γ'_j in such a manner that

$$\gamma = \gamma'_1 * \gamma'_2, \tag{4.1}$$

and the distributions γ'_j are concentrated on the subgroup $p(\mathbb{R}^l)$. It follows from Corollary 2.5 and (4.1) that $\mu = p^{-1}(\gamma) = p^{-1}(\gamma'_1) * p^{-1}(\gamma'_2)$. By Theorem 2.18, $p^{-1}(\gamma'_j) \in \Gamma(\mathbb{R}^l)$. This implies that $\gamma'_j = p(p^{-1}(\gamma'_j)) \in \Gamma(X)$. Thus in the case when γ is a continuous monomorphic image of a Gaussian distribution in a finite-dimensional linear space, the lemma is proved.

Let $\gamma = p(\mu)$, where $p\colon \mathbb{R}^{\aleph_0} \mapsto X$ is a continuous monomorphism, $\mu \in \Gamma(\mathbb{R}^{\aleph_0})$. To prove the lemma in this case we note that Theorem 2.18 implies directly that the corresponding statement is also true for the group $\mathbb{R}^{\aleph_0}$. In all other respects the proof remains the same. □

In connection with Lemma 4.4 we mention the following assertion.

Proposition 4.5. *Let X be a connected group of finite dimension l. Assume that $\gamma \in \Gamma^s(X)$ and γ has only Gaussian factors. Then there exist a finite-dimensional linear space G, a Gaussian distribution $\mu \in \Gamma(G)$, and a continuous monomorphism $p\colon G \mapsto X$ such that $\gamma = p(\mu)$.*

Proof. Let homomorphisms $\pi\colon Y \mapsto \mathbb{R}^l$ and $p\colon X \mapsto \mathbb{R}^l$ be the same as in the proof of Proposition 3.8. By Proposition 3.8, $\gamma = p(\mu)$, where $\mu \in \Gamma(\mathbb{R}^l)$, and the characteristic functions of the distributions μ and γ are of the form

$$\begin{aligned} \hat{\mu}(s) &= \exp\{-\langle As, s\rangle\}, \qquad s \in \mathbb{R}^l, \\ \hat{\gamma}(y) &= \exp\{-\langle A\pi y, \pi y\rangle\}, \quad y \in Y, \end{aligned} \tag{4.2}$$

where $A = (\alpha_{ij})_{i,j=1}^l$ is a symmetric positive semidefinite matrix. The matrix A defines a linear operator $A\colon \mathbb{R}^l \mapsto \mathbb{R}^l$, i.e., a continuous homomorphism of the group $\mathbb{R}^l$

into itself. Set $G = \sigma(\mu)$. Then G coincides with the annihilator $G = A(\mathbb{R}^l, \operatorname{Ker} A)$. We will verify that $G \cap \operatorname{Ker} p = \{0\}$. Thus the desired statement will be proved.

The kernel $\operatorname{Ker} p$ is a closed subgroup of $\mathbb{R}^l$. By Theorem 1.16.1, $\operatorname{Ker} p = F \times S$, where $F \cong \mathbb{R}^n$, $S \cong \mathbb{Z}^m$. We note that in fact $\operatorname{Ker} p = S$. Indeed, if $\lambda t_0 \in \operatorname{Ker} p$ for some $t_0 \in \operatorname{Ker} p$ and all $\lambda \in \mathbb{R}$, then

$$(\lambda t_0, \pi y) = (p(\lambda t_0), y) = 1$$

for all $y \in Y$. But this is impossible because the subspace generated by the image $\pi(Y)$ coincides with $\mathbb{R}^l$ by construction. Thus $\operatorname{Ker} p = S \cong \mathbb{Z}^m$. Assume that there exists an element $a \in G \cap \operatorname{Ker} p$, $a \neq 0$. Since $\operatorname{Ker} p \cong \mathbb{Z}^m$, we can assume that the element a is chosen in such a way that $\lambda a \notin G \cap \operatorname{Ker} p$ for all $\lambda \in (0, 1)$. Note now that the bilinear form $\langle A\,\cdot, \cdot\rangle$ defines a scalar product on the factor space $\mathbb{R}^l / \operatorname{Ker} A$, and $\langle \cdot, a\rangle$ is a linear functional on $\mathbb{R}^l / \operatorname{Ker} A$. This implies that the inequality

$$\langle As, s\rangle \geq \varepsilon \langle s, a\rangle^2, \quad s \in \mathbb{R}^l, \tag{4.3}$$

holds for some $\varepsilon > 0$. Consider a Gaussian distribution μ_1 on the group $\mathbb{R}^l$ with the characteristic function

$$\hat{\mu}_1(s) = \exp\{-\varepsilon \langle s, a\rangle^2\}, \quad s \in \mathbb{R}^l. \tag{4.4}$$

We deduce from (4.2) and (4.3) that μ_1 is a factor of the distribution μ. Hence the distribution $\gamma_1 = p(\mu_1)$ is a factor of the distribution γ. Moreover (4.4) implies that $H = \sigma(\mu_1) = \{t \in \mathbb{R}^l : t = \lambda a, \lambda \in \mathbb{R}\}$, i.e., $H \cong \mathbb{R}$. Since $p(a) = 0$ and $p(\lambda a) \neq 0$ for all $\lambda \in (0, 1)$, we have $p(H) \cong \mathbb{T}$. It follows from 4.1 that the Gaussian distribution γ_1 has non-Gaussian factors. Hence the distribution γ also has non-Gaussian factors. The obtained contradiction proves that $G \cap \operatorname{Ker} p = \{0\}$. Thus Proposition 4.5 is proved. □

Now we will prove the main theorem of this section.

Theorem 4.6. *Every Gaussian distribution on a group X has only Gaussian factors if and only if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$.*

Proof. If a group X contains a subgroup K topologically isomorphic to the circle group $\mathbb{T}$, then a Gaussian distribution on K may be considered as a Gaussian distribution on X. Hence the necessity follows directly from 4.1.

Let us prove sufficiency. Assume that a group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, and $\gamma \in \Gamma(X)$. Without loss of generality, we may suppose that $\gamma \in \Gamma^s(X)$. It follows from Proposition 3.6 that $\sigma(\gamma) = G$, where G is a connected subgroup of X, i.e., $\gamma \in \Gamma^s(G)$, and the group G contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Hence we can assume from the beginning that X is a connected group.

If $\dim X = l < \infty$, then define the homomorphism $\pi : Y \mapsto \mathbb{R}^l$ by formula (3.11) and put $p = \tilde{\pi}$. By Proposition 3.8 we have $\gamma = p(\mu)$, where $\mu \in \Gamma(\mathbb{R}^l)$. Since the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Lemma 4.3 the subgroup $\pi(Y)$ is dense in $\mathbb{R}^l$. By 1.13 (b) this yields that p is a monomorphism. Applying Lemma 4.4 we obtain that the distribution $\gamma = p(\mu)$ has only Gaussian factors.

If $\dim X = \aleph_0$, then define the homomorphism $\pi : Y \mapsto \mathbb{R}^{\aleph_0 *}$ by formula (3.21). Define the homomorphism $p \colon \mathbb{R}^{\aleph_0} \mapsto X$ by the equality $(pt, y) = (t, \pi y)$ for all $t \in \mathbb{R}^{\aleph_0}$, $y \in Y$, and argue as in the previous case. Instead of Proposition 3.8 we use Proposition 3.11. By Lemma 4.2 the subgroup $\pi(Y)$ is dense in $\mathbb{R}^{\aleph_0 *}$. In contrast to the case when $\dim X < \infty$ we can not apply 1.13 (b) and conclude that p is a monomorphism because the groups $\mathbb{R}^{\aleph_0}$ and $\mathbb{R}^{\aleph_0 *}$ are not locally compact. We prove this fact directly. Applying Lemma 4.4 we obtain that the distribution $\gamma = p(\mu)$ has only Gaussian factors. □

Corollary 4.7. *Let a group Y contain no closed subgroup H such that $Y/H \cong \mathbb{Z}$. Let $\varphi(y)$ be a continuous nonnegative function on Y satisfying equation* 2.16 (ii). *Let $f_j(y)$ be continuous normalized positive definite functions on the group Y such that*

$$\exp\{-\varphi(y)\} = f_1(y) f_2(y), \quad y \in Y.$$

Then

$$f_j(y) = (x_j, y) \exp\{-\varphi_j(y)\}, \quad y \in Y,$$

where $x_j \in X$, and $\varphi_j(y)$ are continuous nonnegative functions on Y satisfying equation 2.16 (ii).

Proof. By Theorem 1.9.2 the following conditions are equivalent:

(i) the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, and

(ii) the group Y contains no closed subgroup H such that $Y/H \cong \mathbb{Z}$ □

Remark 4.8. It should be noted that in the class of infinitely divisible distributions a Gaussian distribution on a group X has only Gaussian factors. To prove this assume that

$$\gamma_0 = \gamma_1 * \gamma_2, \tag{4.5}$$

where $\gamma_0 \in \Gamma(X)$, and γ_j are infinitely divisible distributions. Let the distribution γ_0 have the representation $(x_0, 0, \varphi_0)$, and the distributions γ_j have the representations (x_j, F_j, φ_j) (see 2.16). We deduce from (4.5) that the distribution γ_0 has two representations $(x_0, 0, \varphi_0)$ and $(x_1 + x_2, F_1 + F_2, \varphi_1 + \varphi_2)$. The uniqueness of the function φ in the Lévy–Khinchin formula implies that $\varphi_0 = \varphi_1 + \varphi_2$. Hence the infinitely divisible distributions with the representations $(x_1, F_1, 0)$ and $(x_2, F_2, 0)$ are factors of the degenerate distribution E_{x_0}. This implies that the measures F_j are degenerated at zero, i.e., $\gamma_j \in \Gamma(X)$.

We describe now the groups X on which every Gaussian distribution has non-Gaussian factors. Taking into account Proposition 3.6 we may assume that X is a connected group.

Proposition 4.9. *Let X be a connected group. Every Gaussian distribution on the group X such that $\sigma(\gamma) = X$ has non-Gaussian factors if and only if X is topologically isomorphic to the group*

$$\text{(i)} \quad \mathbb{R}^m \times \mathbb{T}.$$

Proof. By Theorem 1.11.2, the group X is topologically isomorphic to the group $\mathbb{R}^m \times K$, where $m \geq 0$, and K is a compact connected group. Assume that X is not topologically isomorphic to a group of the form (i). Then K is not topologically isomorphic to the circle group $\mathbb{T}$. By Theorem 1.7.1, $Y \cong \mathbb{R}^m \times D$, where $D = K^*$. By Theorems 1.6.1 and 1.6.2 D is a discrete torsion-free group. Moreover by Theorem 1.6.3, if $\dim K = l$ then $r(D) = l$, and if $\dim K = \aleph_0$ then $r(D) = \aleph_0$. Denote by π either the homomorphism $\pi\colon D \mapsto \mathbb{R}^l$ defined by (3.6) if $\dim K = l$, or the homomorphism $\pi\colon D \mapsto \mathbb{R}^{\aleph_0 *}$ defined by (3.15) if $\dim K = \aleph_0$. If $\dim K = l$, then denote by $A = \{a_1, \dots, a_l\}$ a finite independent set of real numbers. If $\dim K = \aleph_0$, then denote by $A = \{a_1, \dots, a_n, \dots\}$ an infinite independent set of real numbers. Let either $a = (a_1, \dots, a_l)$ if $\dim K = l$, or $a = (a_1, \dots, a_n, \dots)$ if $\dim K = \aleph_0$. Since $K \not\cong \mathbb{T}$, we have $D \not\cong \mathbb{Z}$. This implies that the set $B = \{t \in \mathbb{R} : t = \langle \pi d, a \rangle,\ d \in D\}$ is dense in $\mathbb{R}$.

Denote by $y = (s, d)$, $s = (s_1, \dots, s_m) \in \mathbb{R}^m$, $d \in D$ elements of the group Y. Consider the continuous homomorphism $\pi_1\colon Y \mapsto \mathbb{R}^{m+1}$ defined by

$$\pi_1(s, d) = (s, \langle \pi d, a \rangle).$$

Set $p_1 = \tilde{\pi}_1$, $p_1\colon \mathbb{R}^{m+1} \mapsto X$. It follows from the independence of the set A that π_1 is a monomorphism, and hence by 1.13 (b) the image $p_1(\mathbb{R}^{m+1})$ is dense in X. On the other hand, since the set B is dense in $\mathbb{R}$, the set $\pi_1(Y)$ is dense in $\mathbb{R}^{m+1}$. Then by 1.13 (b), p_1 is a monomorphism. Let μ be a Gaussian distribution on the group $\mathbb{R}^{m+1}$ with the characteristic function

$$\hat{\mu}(s_1, \dots, s_m, s_{m+1}) = \exp\{-(s_1^2 + \cdots + s_m^2 + s_{m+1}^2)\}.$$

By Proposition 2.10 the characteristic function of the distribution $\gamma = p_1(\mu)$ is of the form

$$\hat{\gamma}(y) = \hat{\gamma}(s_1, \dots, s_m, d) = \exp\{-(s_1^2 + \cdots + s_m^2 + \langle \pi d, a \rangle^2)\}.$$

Put $\gamma = p_1(\mu)$. Since $\overline{p_1(\mathbb{R}^{m+1})} = X$, we have $\sigma(\gamma) = X$. Taking into account that p_1 is a monomorphism, by Lemma 4.4 the distribution γ has only Gaussian factors. The necessity is proved.

Let us prove sufficiency. Let $X = \mathbb{R}^m \times \mathbb{T}$. Then $Y \cong \mathbb{R}^m \times \mathbb{Z}$. Denote by $y = (s, k)$, $s \in \mathbb{R}^m$, $k \in \mathbb{Z}$, elements of the group Y. Without loss of generality, we may assume that γ is a symmetric distribution, i.e., the characteristic function $\hat{\gamma}(y)$ is of the form $\hat{\gamma}(y) = \exp\{-\varphi(y)\}$, where $\varphi(y)$ is a continuous nonnegative function on Y satisfying equation 2.16 (ii). It follows from the proof of Proposition 3.8 that the function $\varphi(y) = \varphi(s_1, \dots, s_m, k)$ has the representation

$$\varphi(s_1, \dots, s_m, k) = \langle As, s \rangle + 2k\langle \beta, s \rangle + bk^2,$$

where $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$, $A = (\alpha_{ij})_{i,j=1}^m$ is a symmetric positive semidefinite matrix, $\beta \in \mathbb{R}^m$, $b \geq 0$. Since $\sigma(\gamma) = X$, it follows from Proposition 2.13 that $\varphi(y) = 0$ if and only if $y = 0$. This implies that the inequality

$$\langle As, s\rangle + 2s_{n+1}\langle \beta, s\rangle + bs_{n+1}^2 > 0 \tag{4.6}$$

holds for all $(s, s_{n+1}) \in \mathbb{R}^{m+1}$, $(s, s_{n+1}) \neq 0$.

It follows from (4.6) that there exists $\varepsilon > 0$ such that the inequality

$$\langle As, s\rangle + 2s_{n+1}\langle \beta, s\rangle + bs_{n+1}^2 \geq \varepsilon(s_1^2 + \cdots + s_m + s_{m+1})$$

is valid for all $(s, s_{n+1}) \in \mathbb{R}^{m+1}$. For this reason the distribution γ has a factor γ_1 with the characteristic function

$$\hat{\gamma}_1(y) = \hat{\gamma}_1(s_1, \ldots, s_m, k) = \exp\{-\varepsilon k^2\}.$$

Since $\sigma(\gamma_1) = \mathbb{T}$ and $\gamma_1 \in \mathrm{M}^1(\mathbb{T})$, by 4.1 the distribution γ_1 has non-Gaussian factors. Hence the distribution γ also has non-Gaussian factors. □

Proposition 4.9 implies directly the following statement.

Corollary 4.10. *Each nondegenerate Gaussian distribution on a group X has non-Gaussian factors if and only if X is topologically isomorphic to the circle group $\mathbb{T}$.*

Remark 4.11. Let X be a connected group. The reasoning given in the proof of necessity of Proposition 4.9 gives us one more construction of a Gaussian distribution $\gamma \in \Gamma(X)$ such that $\sigma(\gamma) = X$ (compare with Remark 3.12).

5 Polynomials on locally compact Abelian groups and the Marcinkiewicz theorem

Let X be a second countable locally compact Abelian group, Y be its character group. According to the classical Marcinkiewicz theorem, if γ is a distribution on the real line and the characteristic function $\hat{\gamma}(s)$ is of the form $\hat{\gamma}(s) = \exp\{P(s)\}$, where $P(s)$ is a polynomial, then γ is a Gaussian distribution. In the first part of this section we obtain the canonical representation for an algebraic polynomial on an arbitrary Abelian group without the assumption of local compactness. Then we study the properties of continuous polynomials on locally compact Abelian groups and describe such groups X for which the Marcinkiewicz theorem holds.

Lemma 5.1. *Let R be a commutative ring with identity 1 and $R[x_1, \ldots, x_n]$ be the polynomial ring over R. Then there exists a nonnegative integer p such that the polynomial*

$$(n!)^{2^p} \prod_{i=1}^{n} (x_i - 1)$$

belongs to the ideal J generated by the polynomials

$$(x_{i_1}x_{i_2}\dots x_{i_k}-1)^n, \quad 1\le i_1<i_2<\dots<i_k\le n.$$

In other words, there exist polynomials $P_{i_1,\dots,i_k}(x_1,\dots,x_n)\in R[x_1,\dots,x_n]$ such that

$$\text{(i)}\quad (n!)^{2^p}\prod_{i=1}^{n}(x_i-1)=\sum_{1\le i_1<i_2<\dots<i_k\le n}P_{i_1,\dots,i_k}(x_1,\dots,x_n)(x_{i_1}x_{i_2}\dots x_{i_k}-1)^n.$$

Proof. We first verify the identity

$$\sum_{k=1}^{n}(-1)^k\sum_{1\le i_1<i_2<\dots<i_k\le n}(x_{i_1}x_{i_2}\dots x_{i_k}-1)^n=\sum_{m=1}^{n}(-1)^mC_n^m\prod_{i=1}^{n}(x_i^m-1). \tag{5.1}$$

For this purpose we remove parentheses in both sides of (5.1). Each of the obtained expressions contains only the monomial of the form

$$(x_{i_1}x_{i_2}\dots x_{i_k})^m, \tag{5.2}$$

where $1\le i_1<i_2<\dots<i_k\le n$ and $0\le m\le n$. If $m>0$, then the coefficients of monomial (5.2) on both sides of (5.1) is equal to

$$(-1)^k(-1)^{n-m}C_n^m.$$

It remains to verify the equality of constant terms in both sides of (5.1). Taking into account what we have already proved we know that the difference of the left-hand side and the right-hand side of (5.1) is a constant. Let us verify that this constant is equal to zero. Substitute $x_1=x_2=\dots=x_n=1$ into (5.1). Then both sides of (5.1) are equal to zero. Hence identity (5.1) is proved.

Let P be the polynomial which is equal to each side of identity (5.1). It follows from (5.1) that $P\in J$ and P is divided by $(x_1-1)(x_2-1)\dots(x_n-1)$. We have

$$P=Q\prod_{i=1}^{n}(x_i-1), \tag{5.3}$$

where

$$Q=\sum_{m=1}^{n}(-1)^mC_n^m\prod_{i=1}^{n}\Big(\sum_{k=0}^{m-1}x_i^k\Big).$$

This formula implies that

$$Q(1,\dots,1)=\sum_{m=1}^{n}(-1)^mC_n^m m^n=(-1)^n n! \tag{5.4}$$

(we used here the well-known combinatorial identity, see for instance [48], § 12, Exercise 16).

Let us introduce new variables $y_i = x_i - 1$, $i = 1, 2, \dots, n$. We deduce from (5.3) and (5.4) that

$$P = (-1)^n y_1 y_2 \dots y_n (n! - \widetilde{Q}), \tag{5.5}$$

where $\widetilde{Q}$ is a polynomial in the variables y_i with zero constant term. Multiplying both sides of (5.5) by

$$S = \prod_{k=0}^{p-1} \left[(n!)^{2^k} + (\widetilde{Q})^{2^k}\right],$$

we get

$$(-1)^n SP = y_1 y_2 \dots y_n \left[(n!)^{2^p} - (\widetilde{Q})^{2^p}\right].$$

It follows from this that

$$(n!)^{2^p} y_1 y_2 \dots y_n = (-1)^n SP + y_1 y_2 \dots y_n (\widetilde{Q})^{2^p}. \tag{5.6}$$

Since the polynomial $\widetilde{Q}$ has zero constant term, we can choose p so large that every term in the expansion of $(\widetilde{Q})^{2^p}$ is divisible by at least one of the polynomials y_i^{n-1}, $i = 1, 2, \dots, n$. Since $P \in J$ and, obviously, $y_i^n \in J$ for all $i = 1, 2, \dots, n$, formula (5.6) implies the inclusion $(n!)^{2^p} y_1 y_2 \dots y_n \in J$. This is equivalent to (i). □

5.2 Application of Lemma 5.1 to finite difference operators. Let Y be an Abelian group, $f(y)$ be a function on Y, and h be an arbitrary element of Y. Denote by E_h the shift operator

$$\mathrm{E}_h f(y) = f(y + h).$$

Then

$$\Delta_h = \mathrm{E}_h - \mathrm{E}_0.$$

Denote by $\mathcal{E}$ the set of finite sums of the form

$$\mathcal{E} = \left\{\sum m_i \mathrm{E}_{u_i} : m_i \in \mathbb{Z},\, u_i \in Y\right\}.$$

It is obvious that $\mathcal{E}$ is a commutative ring with identity. Let $t_1, \dots, t_n \in Y$ be fixed elements and let $P(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$. The mapping

$$P(x_1, \dots, x_n) \mapsto P(\mathrm{E}_{t_1}, \dots, \mathrm{E}_{t_n})$$

is a homomorphism from $\mathbb{Z}[x_1, \dots, x_n]$ to $\mathcal{E}$. Substituting E_{t_i} for x_i into formula 5.1 (i) we arrive at the identity

$$(n!)^{2^p} \Delta_{t_1} \Delta_{t_2} \dots \Delta_{t_n} = \sum_k m_k \mathrm{E}_{u_k} \Delta_{v_k}^n, \tag{5.7}$$

where $m_k \in \mathbb{Z}$, and $u_k, v_k \in Y$ depend on $t_1, \dots, t_n$. Identity (5.7) implies directly the following proposition.

Proposition 5.3. *Let Y be an Abelian group, $f(y)$ be a function on Y. If the function $f(y)$ satisfies the equation*

$$\Delta_h^n f(y) = 0, \quad y, h \in Y,$$

then $f(y)$ satisfies the equation

$$\text{(i)} \quad \Delta_{h_1}\Delta_{h_2}\dots\Delta_{h_n} f(y) = 0, \quad y, h_1, h_2, \dots, h_n \in Y.$$

5.4 n-additive functions. Let Y be an Abelian group. A function $g(y_1, y_2, \dots, y_n)$ defined on Y^n is said to be *n-additive* if the equality

$$\begin{aligned} &g(y_1, y_2, \dots, y_k' + y_k'', \dots, y_n) \\ &\qquad = g(y_1, y_2, \dots, y_k', \dots, y_n) + g(y_1, y_2, \dots, y_k'', \dots, y_n) \end{aligned}$$

holds for all $k = 1, 2, \dots, n$. It is convenient in what follows to call the constants by 0-additive functions.

A function $g(y_1, y_2, \dots, y_n)$ is called *symmetric* if

$$g(y_1, y_2, \dots, y_n) = g(y_{i_1}, y_{i_2}, \dots, y_{i_n})$$

for all permutations $i_1, i_2, \dots, i_n$ of the sequence $1, 2, \dots, n$. For a given function $g(y_1, y_2, \dots, y_n)$ we define the function $g^*(y)$ by the formula

$$g^*(y) = g(y, y, \dots, y).$$

Let us verify that if $g(y_1, y_2, \dots, y_n)$ is a symmetric n-additive function, then the equality

$$\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_p} g^*(y) = \begin{cases} 0 & \text{if } p > n, \\ n!g(u_1, u_2, \dots, u_n) & \text{if } p = n \end{cases} \tag{5.8}$$

holds.

1. First we consider the case when $p > n$. If $n = 1$, then $g^* = g$, and in view of additivity property of the function $g(y)$ we have

$$\Delta_{u_1}\Delta_{u_2} g^*(y) = \Delta_{u_1}\Delta_{u_2} g(y) = 0.$$

Next we argue by induction. We define the function $g_{k,u}(y_1, y_2, \dots, y_k)$ by the function $g(y_1, y_2, \dots, y_n)$ via $g_{k,u}(y_1, y_2, \dots, y_k) = g(y_1, y_2, \dots, y_k, u, \dots, u)$. Making use of the symmetry and additivity properties of the function $g(y_1, y_2, \dots, y_n)$ we find

$$\begin{aligned} \Delta_{u_p} g^*(y) &= g^*(y + u_p) - g^*(y) \\ &= g(y + u_p, \dots, y + u_p) - g(y, \dots, y) \\ &= \sum_{k=0}^{n-1} C_n^k g(\underbrace{y, \dots, y}_{k}, \underbrace{u_p, \dots, u_p}_{n-k}) = \sum_{k=0}^{n-1} C_n^k g^*_{k,u_p}(y), \end{aligned}$$

i.e.,

$$\Delta_{u_p} g^*(y) = \sum_{k=0}^{n-1} C_n^k g_{k,u_p}^*(y). \tag{5.9}$$

By the induction hypothesis we may assume that the equality

$$\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{p-1}} g_{k,u_p}^*(y) = 0 \tag{5.10}$$

holds for all $k = 0, 1, \dots, n-1$. Applying the operator $\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{p-1}}$ to both sides of equality (5.9) and using (5.10) we obtain

$$\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_p} g^*(y) = 0.$$

Thus the first part of (5.8) is proved.

2. Now let $p = n$. If $n = 1$, then $g^* = g$ and we have

$$\Delta_{u_1} g^*(y) = \Delta_{u_1} g(y) = g(y + u_1) - g(y) = g(u_1).$$

Again we argue by induction. Formula (5.9) is also true for $p = n$, namely

$$\Delta_{u_n} g^*(y) = \sum_{k=0}^{n-1} C_n^k g_{k,u_n}^*(y). \tag{5.11}$$

Applying the operator $\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{n-1}}$ to both sides of equality (5.11) and using the result proved in case 1 we get

$$\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_n} g^*(y) = n\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{n-1}} g_{n-1,u_n}^*(y). \tag{5.12}$$

By the induction hypothesis we have

$$\begin{aligned}\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{n-1}} g_{n-1,u_n}^*(y) &= (n-1)! g_{n-1,u_n}(u_1, \dots, u_{n-1}) \\ &= (n-1)! g(u_1, \dots, u_n).\end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.12) we prove the second part of (5.8).

Now we can find the canonical representation for a polynomial on an Abelian group Y. We recall that by a polynomial on Y we understand a function $f(y)$ satisfying the equation

$$\Delta_h^{n+1} f(y) = 0, \quad y, h \in Y, \tag{5.14}$$

for some n.

Theorem 5.5. *Let Y be an Abelian group, $f(y)$ be a function on Y satisfying equation (5.14). Then there exist symmetric k-additive functions $g_k(y_1, y_2, \dots, y_k)$, $k = 0, 1, \dots, n$, such that*

$$\text{(i)} \quad f(y) = \sum_{k=0}^{n} g_k^*(y), \quad y \in Y,$$

where $g_k^*(y) = g_k(y, \dots, y)$.

Proof. Let the function $f(y)$ satisfy equation (5.14). By Proposition 5.3, $f(y)$ satisfies the equation

$$\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_{n+1}}f(y)=0,\quad y,u_1,u_2,\dots,u_{n+1}\in Y. \tag{5.15}$$

It means that the function $\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_n}f(y)$ does not depend on y, i.e., it is a constant. Define the function $g_n(u_1,u_2,\dots,u_n)$ by the formula

$$g_n(u_1,u_2,\dots,u_n)=\frac{1}{n!}\Delta_{u_1}\Delta_{u_2}\dots\Delta_{u_n}f(y). \tag{5.16}$$

It is obvious that the function $g_n(u_1,u_2,\dots,u_n)$ is symmetric. We will verify that the function $g_n(u_1,u_2,\dots,u_n)$ is n-additive. Taking into account (5.15) we get

$$\begin{aligned}
&n![g_n(u_1'+u_1'',u_2,\dots,u_n)-g_n(u_1',u_2,\dots,u_n)-g_n(u_1'',u_2,\dots,u_n)]\\
&\quad=\Delta_{u_1'+u_1''}\Delta_{u_2}\dots\Delta_{u_n}f(y)-\Delta_{u_1'}\Delta_{u_2}\dots\Delta_{u_n}f(y)-\Delta_{u_1''}\Delta_{u_2}\dots\Delta_{u_n}f(y)\\
&\quad=(\Delta_{u_1'+u_1''}-\Delta_{u_1'}-\Delta_{u_1''})\Delta_{u_2}\dots\Delta_{u_n}f(y)\\
&\quad=\Delta_{u_1'}\Delta_{u_1''}\Delta_{u_2}\dots\Delta_{u_n}f(y)=0.
\end{aligned}$$

Thus the function $g_n(u_1,u_2,\dots,u_n)$ is additive in the variable u_1. In view of symmetry of $g_n(u_1,u_2,\dots,u_n)$ we obtain that $g_n(u_1,u_2,\dots,u_n)$ is n-additive.

We will prove the theorem by induction. Obviously, the theorem is true for $n=0$. Let $n\ge 1$. Put $h(y)=f(y)-g_n^*(y)$. We have

$$\Delta_u^n h(y)=\Delta_u^n f(y)-\Delta_u^n g_n^*(y).$$

Note that in view of (5.8),

$$\Delta_u^n g_n^*(y)=n!g_n^*(u).$$

On the other hand it follows from (5.16) that

$$n!g_n^*(u)=\Delta_u^n f(y).$$

Hence the equality

$$\Delta_u^n h(y)=0$$

holds for all $y,u\in Y$. Then by the induction hypothesis

$$h(y)=\sum_{k=0}^{n-1}g_k^*(y),\quad y\in Y,$$

where the functions $g_k(y_1,\dots,y_k)$ are symmetric and k-additive. Therefore

$$f(y)=h(y)+g_n^*(y)=\sum_{k=0}^{n}g_k^*(y),\quad y\in Y.$$ □

Remark 5.6. Let Y be a locally compact Abelian group. We give two simple examples of continuous polynomials on Y. Let $l(y)$ be a *continuous real character* of the group Y, i.e., a continuous homomorphism $l\colon Y \mapsto \mathbb{R}$. If $l(y) \not\equiv 0$, then $l(y)$ is a continuous polynomial of degree 1. Let $\varphi(y)$ be a continuous function on Y satisfying equation 2.16 (ii). If $\varphi(y) \not\equiv 0$, then $\varphi(y)$ is a continuous polynomial of degree 2.

It is not difficult to check that for the group $Y = \mathbb{R}^m$ the set of all continuous polynomials, i.e., the set of all continuous functions on Y satisfying equation (5.14) coincides with the set of ordinary polynomials.

We use representation 5.5 (i) to prove the following property of continuous polynomials on a locally compact Abelian group.

Proposition 5.7. *Let Y be a locally compact Abelian group, $f(y)$ be a continuous polynomial on Y. Then*

$$\text{(i)} \quad f(y+h) = f(y)$$

for all $y \in Y$, $h \in b_Y$. In particular, $f(y) = const$ for $y \in b_Y$.

Proof. First we note that if H is a compact Abelian group and $l(h)$ is a continuous additive function on H, then $l(h) \equiv 0$. Indeed, in this case $l(H)$ is a compact subgroup, but $\{0\}$ is the only compact subgroup of the groups $\mathbb{R}$ and $\mathbb{C}$. It follows from this that if a function $A(y_1, y_2, \dots, y_n)$ defined on Y^n is n-additive, then $A(y_1, y_2, \dots, y_n) = 0$ once $y_k \in b_Y$ for some k.

Let $f(y)$ be a continuous polynomial on the group Y. We make use of representation 5.5 (i). It follows from 5.5 (i) that it suffices to prove (i) for the functions $g_k^*(y)$, $k = 1, 2, \dots, n$. We have

$$\begin{aligned} g_k^*(y+h) - g_k^*(y) &= g_k(y+h, \dots, y+h) - g_k(y, \dots, y) \\ &= \sum_{l=1}^{k} C_k^l g_k(\underbrace{h, \dots, h}_{l}, \underbrace{y, \dots, y}_{k-l}) = 0 \end{aligned}$$

in view of what has been said above. □

Corollary 5.8. *Let X be a second countable locally compact Abelian group, $\mu \in$ $\mathrm{M}^1(X)$. Assume that the characteristic function $\hat{\mu}(y)$ is of the form*

$$\hat{\mu}(y) = \exp\{\psi(y)\}, \quad \psi(0) = 0, \ y \in Y,$$

where $\psi(y)$ is a continuous polynomial. Then $\sigma(\mu) \subset c_X$.

Proof. We deduce from 5.7 (i) that $\psi(y) = \psi(0) = 0$ for $y \in b_Y$. Hence $\hat{\mu}(y) = 1$ for $y \in b_Y$. By Proposition 2.13, $\sigma(\mu) \subset A(X, b_Y)$. By Theorem 1.9.3, $A(X, b_Y) = c_X$. Thus $\sigma(\mu) \subset c_X$. □

Remark 5.9. Let $f(y)$ be a continuous polynomial on a locally compact Abelian group Y. By Proposition 5.7 the function $f(y)$ is invariant with respect to the subgroup b_Y. Hence the function $f(y)$ defines a continuous polynomial $\tilde{f}([y])$ on the factor group Y/b_Y by the formula $\tilde{f}([y]) = f(y)$.

To prove the main theorem of this section we need the following lemma.

Lemma 5.10. *Let K be a compact connected Abelian group,* $\dim K = \aleph_0$. *Assume that K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $D = K^*$, and let $\pi: D \mapsto \mathbb{R}^{\aleph_0 *}$ be the monomorphism defined by* (3.15). *Then for arbitrary $a_1, a_2 \in D$ there exists a subgroup $B \subset D$ of finite rank such that $a_1, a_2 \in B$ and $\overline{\pi(B)} \cong \mathbb{R}^l$, where l is a rank of B.*

Proof. First we check that for each n there exists a number p such that $\mathbb{R}^n \subset \overline{\pi(D) \cap \mathbb{R}^p}$. Indeed, choose a set $E = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$ in such a way that the subgroup $M(E) = \{x \in \mathbb{R}^n : x = \sum_{j=1}^k m_j x_j,\ m_j \in \mathbb{Z}\}$ is dense in $\mathbb{R}^n$. By Lemma 4.2, $\overline{\pi(D)} = \mathbb{R}^{\aleph_0 *}$. This implies that for every element x_j, $j = 1, 2, \dots, k$, there exists a sequence $\{u_i^{(j)}\}_{i=1}^\infty \subset \pi(D)$ such that $u_i^{(j)} \to x_j$. By definition of topology on $\mathbb{R}^{\aleph_0 *}$ all elements $\{u_i^{(j)}\}_{i=1}^\infty$, $j = 1, 2, \dots, k$, belong to some $\mathbb{R}^p$ and they converge in $\mathbb{R}^p$. Therefore $x_j \in \overline{\pi(D) \cap \mathbb{R}^p}$, $j = 1, 2, \dots, k$, and hence $\mathbb{R}^n \subset \overline{\pi(D) \cap \mathbb{R}^p}$.

By Theorem 1.16.1 we have $\overline{\pi(D) \cap \mathbb{R}^p} = G \times F$, where $G \cong \mathbb{R}^{t_p}$, $F \cong \mathbb{Z}^{s_p}$, $t_p + s_p = p$. Therefore $\mathbb{R}^n \subset G$. Let $\pi a_1, \pi a_2 \in \pi(D) \cap \mathbb{R}^n$. Put $A = \pi(D) \cap G$ and $B = \pi^{-1}(A)$. Then $\overline{A} = G$. Taking into account that $r(B) = r(A) = t_p$, we see that B is the required subgroup. □

Now we will prove the maim theorem of this section.

Theorem 5.11. *Let X be a second countable locally compact Abelian group, Y be its character group. Assume that X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $\mu \in \mathrm{M}^1(X)$, and let the characteristic function $\hat{\mu}(y)$ be of the form*

$$\text{(i)} \quad \hat{\mu}(y) = \exp\{\psi(y)\}, \quad \psi(0) = 0,\ y \in Y,$$

where $\psi(y)$ is a continuous polynomial. Then $\mu \in \Gamma(X)$.

Proof. By Corollary 5.8 without loss of generality we can suppose that X is a connected group. Then by Theorem 1.11.2, $X \cong R^m \times K$, where $m \geq 0$ and K is a compact connected group. To avoid introducing new notation we will assume that $X = K$. Moreover we restrict ourselves to the proof for the case when $\dim K = \aleph_0$. The case when $\dim K < \infty$ is easier and may be considered analogously. Set $D = K^*$. By Theorems 1.6.1 and 1.6.2, D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = \aleph_0$.

Set $\varphi(y) = -\operatorname{Re}\psi(y)$, $l(y) = \operatorname{Im}\psi(y)$. Obviously, the functions $\varphi(y)$ and $l(y)$ are also polynomial on Y. We will prove that the function $\varphi(y)$ satisfies the equation 2.16 (ii), and the function $l(y)$ satisfies the equation

$$l(u + v) = l(u) + l(v), \quad u, v \in Y. \tag{5.17}$$

Thus we will prove that $\mu \in \Gamma(X)$.

Let $a_1, a_2 \in D$ be arbitrary but fixed elements, let $\pi: D \mapsto \mathbb{R}^{\aleph_0 *}$ be the monomorphism defined by (3.15). Since the group K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Lemma 5.10 there exists a subgroup $B \subset D$ of finite rank l such that $a_1, a_2 \in B$ and $\overline{\pi(B)} \cong \mathbb{R}^l$. Put $G = \overline{\pi(B)}$.

Let $b_1, \dots, b_l$ be a maximal independent system of elements in $\pi(B)$, and $e_1, \dots, e_l$ be the standard basis in $\mathbb{R}^l$. We will construct the homomorphism $\tau: \pi(B) \mapsto \mathbb{R}^l$ just as the monomorphism π was constructed in Proposition 3.8. We have $\tau b_j = e_j$, $j = 1, 2, \dots, l$. Put $h = \tau \circ \pi$, $A = h(B)$. It follows from $\overline{\pi(B)} = G$ that $\overline{A} = \mathbb{R}^l$.

Consider the function $\zeta(a) = \psi(h^{-1}a)$ on the group A. Obviously, $\zeta(a)$ is a polynomial. If we consider the group A in the discrete topology, then by the Bochner theorem $\exp\{\zeta(a)\}$ is a characteristic function. It is easy to see that the theorem will be proved if we prove the following statement.

Let A be a group such that $\mathbb{Z}^l \subset A \subset \mathbb{Q}^l \subset \mathbb{R}^l$, $\overline{A} = \mathbb{R}^l$. Assume that $\zeta(a)$ is a polynomial on the group A such that $\exp\{\zeta(a)\}$ is a characteristic function. Then the functions $\zeta_1(a) = \operatorname{Re} \zeta(a)$ and $\zeta_2(a) = \operatorname{Im} \zeta(a)$ satisfy equations 2.16 (ii) and (5.17) respectively.

Let $\eta(a)$ be an arbitrary polynomial of degree m on the group A. Then the equality

$$\Delta_b^{m+1} \eta(a) = 0$$

is fulfilled for all $a, b \in \mathbb{Z}^l$. It is easily verified that this yields the representation

$$\eta(a) = \sum_{p=0}^{m} \sum_{\|k\|=p} c_k a^k, \tag{5.18}$$

where $k = (k_1, \dots, k_l)$, $\|k\| = k_1 + \dots + k_l$, $a = (n_1, \dots, n_l) \in \mathbb{Z}^l$, $a^k = n_1^{k_1} \dots n_l^{k_l}$. We deduce from (5.18) that if F is a subgroup of $\mathbb{Q}^l$ such that $F \cong \mathbb{Z}^l$, $\mathbb{Z}^l \subset F \subset A$ and $\eta(a) = 0$ for $a \in \mathbb{Z}^l$, then $\eta(a) = 0$ for all $a \in F$.

Represent the group A as a union of an increasing sequence of subgroups A_j:

$$A = \bigcup_{j=1}^{\infty} A_j, \quad A_1 = \mathbb{Z}^l, \quad A_j \subset A_{j+1}, \quad A_j \cong \mathbb{Z}^l, \quad j = 1, 2, \dots.$$

Denote by $\eta(a)$ the restriction of the polynomial $\zeta(a)$ to the subgroup $\mathbb{Z}^l$. We note that the polynomial $\eta(a)$ on the subgroup $\mathbb{Z}^l$ can be represented in the form (5.18). By formula (5.18) the polynomial $\eta(a)$ can be extended to the group $\mathbb{R}^l$, in particular to the group A. Denote by $\tilde{\eta}(a)$ this extension. Consider on the group A the polynomial $\delta(a) = \zeta(a) - \tilde{\eta}(a)$, $a \in A$. It is obvious that $\delta(a) = 0$, $a \in \mathbb{Z}^l$. Then, as has been noted above $\delta(a) = 0$ for all $a \in A_j$, $j = 1, 2, \dots$. Hence $\delta(a) \equiv 0$ on the group A.

From what has been said it follows that the polynomial $\zeta(a)$ on the group A can be represented in the form

$$\zeta(a) = \sum_{p=0}^{m} \sum_{\|k\|=p} c_k a^k, \tag{5.19}$$

where $k = (k_1, \dots, k_l)$, $a = (r_1, \dots, r_l) \in A$, $a^k = r_1^{k_1} \dots r_l^{k_l}$. The polynomial $\zeta(a)$ is extended by formula (5.19) from the group A to $\mathbb{R}^l$. Denote by $\tilde{\zeta}(s)$ this extension. Since $\exp\{\zeta(a)\}$ is a positive definite function on the group A and the group A is dense in $\mathbb{R}^l$, the extended function $\exp\{\tilde{\zeta}(s)\}$, $s \in \mathbb{R}^l$, is a positive definite function on the group $\mathbb{R}^l$. By the Bochner theorem $\exp\{\tilde{\zeta}(s)\}$ is a characteristic function.

Fix $s_0 \in \mathbb{R}^l$. Then the function $\tilde{\zeta}(ts_0)$, $t \in \mathbb{R}$, is a polynomial on $\mathbb{R}$ and $\exp\{\tilde{\zeta}(ts_0)\}$ is a characteristic function on $\mathbb{R}$. By the classical Marcinkiewicz theorem (see e.g. [74], Chapter II, § 5) $\exp\{\tilde{\zeta}(ts_0)\}$ is the characteristic function of a Gaussian distribution on $\mathbb{R}$. We deduce from this that $\exp\{\tilde{\zeta}(s)\}$ is the characteristic function of a Gaussian distribution on the group $\mathbb{R}^l$. Hence the function $\tilde{\zeta}_1(s) = \mathrm{Re}\,\tilde{\zeta}(s)$ satisfies equation 2.16 (ii), and the function $\tilde{\zeta}_2(s) = \mathrm{Im}\,\tilde{\zeta}(s)$ satisfies equation (5.17) on the group $\mathbb{R}^l$. It follows from this that the functions $\zeta_1(a)$ and $\zeta_2(a)$ also satisfy equations 2.16 (ii) and (5.17) on the group A. □

Remark 5.12. The following statement results from the proof of Theorem 5.11. Let $Y = \mathbb{R}^p \times \mathbb{Z}^q$, let $\zeta(y)$ be a continuous polynomial on the group Y of degree m. Then the polynomial $\zeta(y)$ is represented in the form

$$\zeta(y) = \sum_{i=0}^{m} \sum_{\|k\|=i} c_k y^k,$$

where $k = (k_1, \dots, k_p, k_{p+1}, \dots, k_{p+q})$, $y = (s_1, \dots, s_p, n_1, \dots, n_q) \in \mathbb{R}^p \times \mathbb{Z}^q$, $y^k = s_1^{k_1} \dots s_p^{k_p} n_1^{k_{p+1}} \dots n_q^{k_{p+q}}$. To prove this we note that by virtue of continuity, the polynomial $\zeta(y)$ is defined by its restriction to the subgroup $A = \mathbb{Q}^p \times \mathbb{Z}^q \subset \mathbb{R}^p \times \mathbb{Z}^q$. Then we use representation (5.19).

Theorem 5.11 is sharp. Namely, the following result holds.

Proposition 5.13. *Let X be a second countable locally compact Abelian group, Y be its character group. Assume that X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$. Then for every $m > 2$ there exists a distribution $\mu_m \in \mathrm{M}^1(X)$ such that $\mu_m \notin \Gamma(X)$, but the characteristic function $\hat{\mu}_m(y)$ is of the form*

$$\hat{\mu}_m(y) = \exp\{\psi_m(y)\}, \quad \psi_m(0) = 0, \ y \in Y,$$

where $\psi_m(y)$ is a continuous polynomial of degree m.

Proof. Obviously, it suffices to prove the proposition for the circle group $X = \mathbb{T}$. Then $Y \cong \mathbb{Z}$. We will suppose without loss of generality that $Y = \mathbb{Z}$. Consider the function

$$\psi_m(n) = \begin{cases} -n^2 + i n^m, \ n \in \mathbb{Z}, & \text{if } m \text{ is odd}, \\ -n^m, \ n \in \mathbb{Z}, & \text{if } m \text{ is even}, \end{cases}$$

on the group $\mathbb{Z}$. It is clear that $\psi_m(n)$ is a polynomial of degree m on the group $\mathbb{Z}$. Set

$$g_m(n) = \exp\{\psi_m(n)\}, \quad n \in \mathbb{Z}.$$

It is easy to verify that

$$\sum_{n\in\mathbb{Z},n\neq 0} g_m(n) < 1.$$

This implies that

$$\rho_m(t) = 1 + \sum_{n\in\mathbb{Z},n\neq 0} g_m(n)\exp\{-int\} > 0, \quad t \in \mathbb{R}.$$

It is also obvious that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_m(t)\,dt = 1.$$

Let μ_m be the distribution on the circle group $\mathbb{T}$ with density $r_m(e^{it}) = \rho_m(t)$ with respect to $m_{\mathbb{T}}$. Then $\mu_m \notin \Gamma(\mathbb{T})$ and the characteristic function $\hat{\mu}_m(n) = g_m(n)$, $n \in \mathbb{Z}$, has the desired form. □

Remark 5.14. It is easy to see that Proposition 5.13 fails for the circle group $X = \mathbb{T}$ when $m = 2$. It means that the condition $m > 2$ in Proposition 5.13 can not be replaced by the condition $m \geq 2$. Nevertheless the statement of Proposition 5.13 also holds true for $m = 2$, if we assume that a group X contains a subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Obviously, it suffices to prove the statement for the group $X = \mathbb{T}^2$. Then $Y \cong \mathbb{Z}^2$. We will suppose without loss of generality that $Y = \mathbb{Z}^2$. Denote by $y = (m,n)$, $m,n \in \mathbb{Z}$, elements of the group Y. Consider on the group $\mathbb{Z}^2$ the function

$$\psi(m,n) = -a(m^2+n^2) + i\pi mn, \quad (m,n) \in \mathbb{Z}^2.$$

It is clear that $\psi(m,n)$ is a polynomial of degree 2. Choose $a > 0$ in such a way that the inequality

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \exp\{-a(m^2+n^2)\} < 1$$

is fulfilled. It follows from this that the inequality

$$\rho(t,s) = 1 + \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \exp\{\psi(m,n) - i(mt+ns)\} > 0, \quad (t,s) \in \mathbb{R}^2,$$

holds. Also, obviously

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \rho(t,s)\,dt\,ds = 1.$$

Let μ be the distribution on $\mathbb{T}^2$ with density $r(e^{it},e^{is}) = \rho(t,s)$ with respect to $m_{\mathbb{T}^2}$. Then $\mu \notin \Gamma(\mathbb{T}^2)$, and the characteristic function

$$\hat{\mu}(m,n) = \exp\{\psi(m,n)\}, \quad (m,n) \in \mathbb{Z}^2,$$

has the required form.

We will prove below (see Proposition 9.8) that if X is a second countable locally compact Abelian group containing no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$, $\mu \in \mathrm{M}^1(X)$, and the characteristic function $\hat{\mu}(y)$ is represented in the form 5.11 (i), where $\psi(y)$ is a continuous polynomial of degree 2, then $\mu \in \Gamma(X)$.

6 Gaussian distributions in the sense of Urbanik

Let X be a second countable locally compact Abelian group, Y be its character group. Urbanik defined Gaussian distributions on the group X as distributions μ having the following properties: (i) μ is an infinitely divisible distribution; (ii) $y(\mu) \in \Gamma(\mathbb{T})$ for each character $y \in Y$ ([100]). As has been proved in Proposition 3.17 the class of such distributions coincides with the class of Gaussian distributions in the sense of Definition 3.1. In this section we describe completely groups X with the following property: if a distribution μ on X satisfies (ii), then μ satisfies (i), i.e., μ is Gaussian.

Definition 6.1. A distribution γ on a group X is called a *Gaussian distribution in the sense of Urbanik*, if $y(\gamma) \in \Gamma(\mathbb{T})$ for every $y \in Y$.

Denote by $\Gamma_U(X)$ the set of Gaussian distributions in the sense of Urbanik on the group X. We note that for any distribution $\mu \in \mathrm{M}^1(X)$ and for any $y \in Y$ the characteristic function of the distribution $y(\mu)$ has the form

$$\widehat{y(\mu)}(n) = \hat{\mu}(ny), \quad n \in \mathbb{Z}.$$

It follows that

$$\Gamma(X) \subset \Gamma_U(X). \tag{6.1}$$

6.2 Infinitely divisible elements. Let $n \in \mathbb{Z}$, $n \neq 0$. We say that an element $x_0 \in X$ is *divisible by* n if $x_0 \in X^{(n)}$. To say it in other words, an element $x_0 \in X$ is divisible by n if there exists an element $x \in X$ such that $x_0 = nx$. An element $x_0 \in X$ different from zero is called *infinitely divisible*, if it is divisible by infinitely many integers.

To prove the main theorem we need the following lemmas.

Lemma 6.3. *If a group X contains no infinitely divisible elements, then X is a discrete torsion-free group.*

Proof. Let X be an arbitrary locally compact Abelian group. By Theorem 1.11.1, $X \cong \mathbb{R}^m \times G$, where $m \geq 0$, and G contains a compact open subgroup K. Obviously, if X contains no infinitely divisible elements, then $m = 0$, and G is a torsion-free group. In particular, K is a torsion-free group. By Theorem 1.11.4, K is topologically isomorphic to a group of the form

$$(\Sigma_{\boldsymbol{a}})^{\mathfrak{n}} \times \mathop{\mathbf{P}}_{p \in \mathcal{P}} \Delta_p^{\mathfrak{n}_p},$$

where $\boldsymbol{a} = (2, 3, 4, \dots)$, $\mathfrak{n}$ and $\mathfrak{n}_p$ are cardinal numbers. We note that all nonzero elements of the groups $\Sigma_{\boldsymbol{a}}$ and Δ_p are infinitely divisible. Hence $K = \{0\}$, i.e., X is a discrete torsion-free group. □

Lemma 6.4. *On the group $X = \mathbb{R} \times \mathbb{T}$ there exists a distribution μ_0 such that $\mu_0 \in \Gamma_U(X)$, $\mu_0 \notin \Gamma(X)$, and $\hat{\mu}_0(y) > 0$ for all $y \in Y$.*

Proof. We have $Y \cong \mathbb{R} \times \mathbb{Z}$. To avoid introducing new notation we will assume that $Y = \mathbb{R} \times \mathbb{Z}$, and denote by $y = (s, n)$, $s \in \mathbb{R}$, $n \in \mathbb{Z}$, elements of the group Y. Let $a > 0$, $b > 0$, $c > 0$. Consider on the group Y the function

$$\varphi_0(s, n) = \begin{cases} as^2 & \text{if } n = 0, \\ bs^2 + cn^2 & \text{if } n \neq 0. \end{cases} \tag{6.2}$$

Let α_a be a Gaussian distribution on the group X with the characteristic function $\hat{\alpha}_a(s, n) = \exp\{-as^2\}$, and β_c be a Gaussian distribution on the group X with the characteristic function $\hat{\beta}_c(s, n) = \exp\{-cn^2\}$. Obviously, $\sigma(\alpha_a) = \mathbb{R}$, $\sigma(\beta_c) = \mathbb{T}$, i.e., α_a can be considered as a Gaussian distribution on $\mathbb{R}$, and β_c can be considered as a Gaussian distribution on $\mathbb{T}$. Choose a, b and c in such a way that $2\alpha_a \geq \alpha_b$, $2\beta_c \geq m_{\mathbb{T}}$. Put

$$\mu_0 = \alpha_a * m_{\mathbb{T}} - \alpha_b * m_{\mathbb{T}} + \beta_c * \alpha_b. \tag{6.3}$$

It is easy to see that we can transform (6.3) to the form

$$\mu_0 = (\alpha_a - \tfrac{1}{2}\alpha_b) * m_{\mathbb{T}} + (\beta_c - \tfrac{1}{2} m_{\mathbb{T}}) * \alpha_b.$$

This implies that $\mu_0 \in \mathrm{M}^1(X)$. Taking into account 2.7 (c), it follows from (6.2) and (6.3) that

$$\hat{\mu}_0(y) = \exp\{-\varphi_0(y)\}, \quad y \in Y. \tag{6.4}$$

We note that the equality

$$\varphi_0(ky) = k^2 \varphi_0(y), \quad y \in Y, \tag{6.5}$$

holds for all $k \in \mathbb{Z}$. It follows from (6.4) and (6.5) that $\mu_0 \in \Gamma_U(X)$. Choosing $a \neq b$ we obtain that $\mu_0 \notin \Gamma(X)$. □

6.5. Let X be a discrete torsion-free group. An element $x \in X$ is said to be *dependent on elements* $x_1, \dots, x_l \in X$ if there exist $n, n_1, \dots, n_l \in \mathbb{Z}$ such that $nx = n_1 x_1 + \cdots + n_l x_l$. Denote by L_x the set of elements depending on x. Obviously, L_x is a subgroup and L_x is isomorphic to some subgroup of $\mathbb{Q}$.

We can prove now the main theorem of this section.

Theorem 6.6. *For a group X the equality*

$$\Gamma(X) = \Gamma_U(X)$$

holds if and only if one of the following conditions is satisfied:

(i) *for any closed subgroup $B \subset Y$, $B \neq Y$, the factor group Y/B contains an infinitely divisible element;*

(ii) *the group $\mathbb{Z}$ is topologically isomorphic to the factor group of Y by the subgroup b_Y of all compact elements in Y.*

Proof. Necessity. Assume that there exists a closed subgroup $B \subset Y$ such that the factor group $Y/B = H$ contains no infinitely divisible elements. By Lemma 6.3, in this case H is a discrete torsion-free group. Two cases are possible.

1. H is not topologically isomorphic to the group $\mathbb{Z}$. It follows from this that the rank $r(H) > 1$. Indeed, if $r(H) = 1$, then the group H is isomorphic to some subgroup A of the group $\mathbb{Q}$. Since A is not isomorphic to $\mathbb{Z}$, all nonzero elements of A are infinitely divisible. Hence all nonzero elements of H are also infinitely divisible. This contradicts the condition.

For each $h \in H$, $h \neq 0$, let L_h be the subgroup of H of all elements depending on h. By condition the subgroup L_h contains no infinitely divisible elements and $r(L_h) = 1$. This implies that $L_h \cong \mathbb{Z}$ for all $h \in H$, $h \neq 0$. Since the group X is second countable, the group H is countable. Obviously, the group H is at most a countable union of different subgroups L_{h_k} such that $L_{h_i} \cap L_{h_j} = \{0\}, i \neq j$. Denote by h'_k a generator of the group L_{h_k}. We have $H = \bigcup_k L_{h_k}$, and each element $h \in H$, $h \neq 0$, can be uniquely represented in the form $h = mh'_k$, $m \in \mathbb{Z}$, $h'_k \in L_{h_k}$. Define on the group H the function

$$\varphi_0(h) = \begin{cases} a_k m^2 & \text{if } h = mh'_k,\ h \neq 0, \\ 0 & \text{if } h = 0, \end{cases}$$

where the numbers a_k are chosen in such a way that

$$\sum_k \sum_{m \neq 0} \exp\{-a_k m^2\} < 1. \tag{6.6}$$

Set $G = H^*$. It follows from (6.6) that we can define the continuous function $\rho_0(g)$ on the group G by the formula

$$\rho_0(g) = 1 + \sum_{h \in H, h \neq 0} \exp\{-\varphi_0(h)\}\overline{(g,h)}, \quad g \in G.$$

It is obvious that $\rho_0(g) > 0$ and $\int_G \rho_0(g) dm_G(g) = 1$. Let μ_0 be the distribution on G with density $\rho_0(g)$ with respect to m_G. Then the characteristic function $\hat{\mu}_0(h)$ is of the form

$$\hat{\mu}_0(h) = \exp\{-\varphi_0(h)\}, \quad h \in H.$$

Since $r(H) > 1$, the numbers a_k can be chosen in such a way that $\mu_0 \notin \Gamma(G)$. On the other hand, $\mu_0 \in \Gamma_U(G)$ because the function $\varphi(h)$ satisfies the equation

$$\varphi(nh) = n^2\varphi(h), \quad h \in H,$$

for all $n \in \mathbb{Z}$. By Theorem 1.9.2, $G \cong A(X, B)$. Therefore we can consider μ_0 as a distribution on $A(X, B)$. Hence μ_0 can be considered as a distribution on the group X. In the case 1 the necessity is proved.

2. H is topologically isomorphic to the group $\mathbb{Z}$. By Theorem 1.17.2, $Y = B \times L$, where $L \cong \mathbb{Z}$. We note that the subgroup B contains a noncompact element, because otherwise condition (ii) would be satisfied. Put $G = c_{B^*}$. By Theorem 1.9.3, $G \neq \{0\}$. Taking into consideration Theorem 1.7.1, we obtain that the group X contains a subgroup F such that $F \cong G \times \mathbb{T}$. Since the group G is connected, by Theorem 1.11.2, $G \cong \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact connected group. If $m > 0$, then the group X contains a subgroup $M \cong \mathbb{R} \times \mathbb{T}$. Applying Lemma 6.4 we construct a distribution $\mu_0 \in \mathrm{M}^1(X)$ such that $\mu_0 \in \Gamma_U(X)$ and $\mu_0 \notin \Gamma(X)$. If $m = 0$, then $F \cong K \times \mathbb{T}$. Assume for definiteness that the group K has infinite dimension. Then $\dim K = \aleph_0$. Set $D = K^*$. Let $\pi \colon D \mapsto \mathbb{R}^{\aleph_0 *}$ be the homomorphism defined by (3.15), i.e., $\pi d = (k_1/k, \dots, k_l/k, 0, \dots)$. Denote by (d, n), $d \in D$, $n \in \mathbb{Z}$, elements of the group $F^* \cong D \times \mathbb{Z}$. Define the homomorphism $\tau \colon F^* \mapsto \mathbb{R} \times \mathbb{Z}$ by the formula $\tau(d, n) = (k_1/k, n)$. Put $q = \tilde{\tau}$, $q \colon \mathbb{R} \times \mathbb{T} \mapsto F$, and $\mu = q(\mu_0)$, where μ_0 is the distribution constructed in Lemma 6.4. By Proposition 2.10, $\hat{\mu}(d, n) = \widehat{q(\mu_0)}(d, n) = \hat{\mu}_0(\tau(d, n)) = \exp\{-\varphi_0(\tau(d, n))\}$. It follows from Lemma 6.4 that $\mu \in \Gamma_U(F)$ and $\mu \notin \Gamma(F)$. Hence $\mu \in \Gamma_U(X)$ and $\mu \notin \Gamma(X)$. The necessity is proved.

Sufficiency. Suppose that $\gamma \in \Gamma_U(X)$. We will verify that $\gamma \in \Gamma(X)$. Assume first that condition (i) is satisfied. In particular it follows from this that any factor group of the group X is not topologically isomorphic to $\mathbb{Z}$. Then by Theorem 1.9.2 the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Taking into account Theorem 4.6 we see that it suffices to verify that the distribution $\nu = \gamma * \bar{\gamma} \in \Gamma(X)$.

Note that it follows from $\gamma \in \Gamma_U(X)$ that $\hat{\gamma}(y) \neq 0$ for all $y \in Y$. Hence by 2.7 (c) and 2.7 (d), $\hat{\nu}(y) = |\hat{\gamma}(y)|^2 > 0$, and the function $\hat{\nu}(y)$ has the representation

$$\hat{\nu}(y) = \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $\varphi(y)$ is a continuous nonnegative function on the group Y satisfying the equation

$$\varphi(ky) = k^2\varphi(y), \quad y \in Y, \tag{6.7}$$

for all $k \in \mathbb{Z}$. We will verify that if condition (i) is satisfied, then the function $\varphi(y)$ satisfies equation 2.16 (ii). Denote by A the set of all elements $y \in Y$ for which there exists a sequence of elements $\{y_n\} \subset Y$ such that $y_n \in Y^{(n)}$ and $\varphi(y - y_n) \to 0$.

We will prove first that A is a subgroup of Y. Let $u, v \in A$ and let $\{u_n\}$ and $\{v_n\}$ be sequences of elements of Y such that $u_n, v_n \in Y^{(n)}$ and

$$\varphi(u - u_n) \to 0, \quad \varphi(v - v_n) \to 0.$$

Applying inequality 2.7 (g) we obtain

$$\begin{aligned} |1 - \hat{\nu}(u - v - u_n + v_n)| &\leq |\hat{\nu}(u - v - u_n + v_n) - \hat{\nu}(v - v_n)| + |1 - \hat{\nu}(v - v_n)| \\ &\leq \sqrt{2}|1 - \hat{\nu}(u - u_n)|^{1/2} + |1 - \hat{\nu}(v - v_n)|. \end{aligned}$$

Let $\{y_n\} \subset Y$ be an arbitrary sequence. It is obvious that $\varphi(y_n) \to 0$ if and only if $\hat{\nu}(y_n) \to 1$. Therefore it follows from the above inequality that $\varphi(u-v-u_n+v_n) \to 0$. It is clear that $u_n - v_n \in Y^{(n)}$. So, $u - v \in A$. Thus we proved that A is an algebraic subgroup of Y.

Now we will verify that the subgroup A is closed. Indeed, let $\{u^{(j)}\} \subset A, u^{(j)} \to u$, and let $\{u_n^{(j)}\}$ be the sequence corresponding to the element $u^{(j)}$. Applying equality 2.7 (g) we obtain

$$\begin{aligned} |1 - \hat{\nu}(u - u_n^{(j)})| &\le |\hat{\nu}(u - u_n^{(j)}) - \hat{\nu}(u^{(j)} - u_n^{(j)})| + |1 - \hat{\nu}(u^{(j)} - u_n^{(j)})| \\ &\le \sqrt{2}|1 - \hat{\nu}(u - u^{(j)})|^{1/2} + |1 - \hat{\nu}(u^{(j)} - u_n^{(j)})|. \end{aligned} \tag{6.8}$$

As has been noted above, $\varphi(y_n) \to 0$ if and only if $\hat{\nu}(y_n) \to 1$. Therefore (6.8) implies the completeness of A.

We will prove that the factor group Y/A contains no infinitely divisible elements. Assume the contrary. Then there exist an element $u_0 \notin A$, an unbounded sequence of numbers $\{p_n\} \subset \mathbb{Z}$, and a sequence $\{y_n\} \subset Y$ such that

$$u_0 - p_n y_n \in A \tag{6.9}$$

for all natural n. Without loss of generality, we assume that the inequality $p_n \ge n^2$ is satisfied. Therefore there exists a sequence of integers $\{q_n\}$ such that

$$\frac{nq_n}{p_n} \to 1. \tag{6.10}$$

It follows from the definition of the set A and (6.9) that there exists an element $v_n \in Y$ such that $v_n \in Y^{(p_n)}$ and $\varphi(u_0 - p_n y_n - v_n)$ is arbitrarily small. Thus there exists a sequence $\{w_n\} \subset Y$ such that $w_n \in Y^{(p_n)}$ and $\varphi(u_0 - w_n) \to 0$. Put

$$u_n = \frac{nq_n w_n}{p_n}, \quad n = 1, 2, \ldots.$$

Applying inequality 2.7 (g) we get

$$\begin{aligned} |1 - \hat{\nu}(u_0 - u_n)| &\le |\hat{\nu}(u_0 - u_n) - \hat{\nu}(u_0 - w_n)| + |1 - \hat{\nu}(u_0 - w_n)| \\ &\le \sqrt{2}|1 - \hat{\nu}(u_n - w_n)|^{1/2} + |1 - \hat{\nu}(u_0 - w_n)|. \end{aligned} \tag{6.11}$$

It follows from $\varphi(u_0 - w_n) \to 0$ that $\hat{\nu}(u_0 - w_n) \to 1$. Taking into account inequality 2.7 (g), we conclude that $\hat{\nu}(w_n) \to \hat{\nu}(u_0)$. Hence

$$\varphi(w_n) \to \varphi(u_0). \tag{6.12}$$

Taking into account (6.7), it follows from (6.10) and (6.12) that

$$\varphi(u_n - w_n) = \varphi\left(\frac{(nq_n - p_n)w_n}{p_n}\right) = \left(\frac{nq_n}{p_n} - 1\right)^2 \varphi(w_n) \to 0.$$

Therefore, $\hat{\nu}(u_n - w_n) \to 1$. Hence we find from (6.11) that $\hat{\nu}(u_0 - u_n) \to 1$, i.e., $\varphi(u_0 - u_n) \to 0$. Since $u_n \in Y^{(n)}$, this contradicts the assumption that $u_0 \notin A$.

Thus we have proved that the equality $Y = A$ follows from condition (i). Let $u, v \in Y$. Then we can find sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n, v_n \in Y^{(n)}$ and $\varphi(u - u_n) \to 0$, $\varphi(v - v_n) \to 0$. We deduce from inequality 2.7 (g) that in this case the convergences

$$\begin{aligned} \varphi(u_n) &\to \varphi(u), & \varphi(v_n) &\to \varphi(v), \\ \varphi(u_n + v_n) &\to \varphi(u + v), & \varphi(u_n - v_n) &\to \varphi(u - v) \end{aligned} \tag{6.13}$$

hold. By the Bochner theorem the function $\exp\{-\varphi(y)\}$ is positive definite. Therefore for all $y_1, y_2, \dots, y_m \in Y$ and $z_1, z_2, \dots, z_m \in \mathbb{C}$, inequality 2.8 (i) is fulfilled, i.e.,

$$\sum_{i,j=1}^{m} \exp\{-\varphi(y_i - y_j)\} z_i \bar{z}_j \geq 0.$$

Putting here $m = 4$, $y_1 = -y_2 = u_n/n$, $y_3 = -y_4 = v_n/n$, $z_1 = z_2 = -z_3 = -z_4 = n$ we obtain

$$\begin{aligned} &[2\exp\{-(4/n^2)\varphi(u_n)\} + 2\exp\{-(4/n^2)\varphi(v_n)\} \\ &\quad - 4\exp\{-(1/n^2)\varphi(u_n - v_n)\} - 4\exp\{-(1/n^2)\varphi(u_n + v_n)\} + 4]n^2 \geq 0. \end{aligned}$$

Proceeding to the limit as $n \to \infty$ and taking into account (6.13) we find

$$4\varphi(u + v) + 4\varphi(u - v) - 8\varphi(u) - 8\varphi(v) \geq 0,$$

or

$$2[\varphi(u) + \varphi(v)] \leq \varphi(u + v) + \varphi(u - v). \tag{6.14}$$

Replacing here u by $u + v$ and v by $u - v$ and taking into consideration (6.7), we get

$$\varphi(u + v) + \varphi(u - v) \leq 2[\varphi(u) + \varphi(v)]. \tag{6.15}$$

It follows from (6.14) and (6.15) that the function $\varphi(y)$ satisfies equation 2.16 (ii), i.e., $\nu \in \Gamma(X)$. So, $\Gamma_U(X) \subset \Gamma(X)$, and hence taking into account (6.1), sufficiency of condition (i) is proved.

To prove the sufficiency of condition (ii) we note that the support of a distribution $\gamma \in \Gamma_U(X)$ is contained in a coset of the connected component of zero c_X of the group X. Indeed, let $\nu = \gamma * \bar{\gamma}$. Taking into account Proposition 2.2, it suffices to show that $\sigma(\nu) \subset c_X$. Let $y_0 \in b_Y$. Since y_0 is a compact element, we have $n_l y_0 \to \tilde{y} \in Y$ for some sequence $n_l \to \infty$. Taking into consideration (6.7), we get

$$\varphi(y_0) = \lim_{n_l \to \infty} \frac{\varphi(\tilde{y})}{n_l^2} = 0.$$

Hence $\varphi(y) = 0$ for all $y \in b_Y$. It follows from Proposition 2.13 that $\sigma(\nu) \subset A(X, b_Y)$. By Theorem 1.9.3, $A(X, b_Y) = c_X$. Hence $\sigma(\nu) \subset c_X$.

Now it is easy to verify the sufficiency of condition (ii). Replace the distribution γ by its shift $\gamma' = \gamma * E_x$ such that $\sigma(\gamma') \subset c_X$. But if the group X satisfies condition (ii), then by Theorems 1.9.2 and 1.9.3, $c_X \cong \mathbb{T}$. Since $\Gamma_U(\mathbb{T}) = \Gamma(\mathbb{T})$ and $\gamma' \in \Gamma_U(c_X)$, we have $\gamma' \in \Gamma(c_X)$. Hence $\gamma \in \Gamma(X)$. So $\Gamma_U(X) \subset \Gamma(X)$. Taking into account (6.1), sufficiency of condition (ii) is also proved. $\square$

Chapter III
The Kac–Bernstein theorem for locally compact Abelian groups

According to the classical Kac–Bernstein theorem, if ξ_1 and ξ_2 are independent random variables and their sum and difference are also independent, then ξ_j are Gaussian. Let X be a second countable locally compact Abelian group. In this chapter we study the distributions of independent random variables ξ_1 and ξ_2 taking values in X and having independent sum and difference. We give a complete description of groups X on which such distributions are invariant with respect to some compact subgroup K of X and, under the natural homomorphism $X \mapsto X/K$, induce Gaussian distributions on the factor group X/K. It holds if and only if the connected component of zero of a group X contains no elements of order 2. Hence if the connected component of zero of a group X contains elements of order 2, then for such groups the following natural problem arises: to describe distributions of independent random variables ξ_j taking values in X and having independent sum and difference. We solve this problem for the group $\mathbb{R} \times \mathbb{T}$ and for $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$. We also study the distributions of independent identically distributed random variables ξ_1 and ξ_2 taking values in a group X and having independent sum and difference (Gaussian distributions in the sense of Bernstein).

7 Locally compact Abelian groups for which the Kac–Bernstein theorem holds

Let X be a second countable locally compact Abelian group, Y be its character group. In this section we describe groups X with the property: if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_1, μ_2 and $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then μ_j are invariant with respect to some compact subgroup K of the group X and under the natural homomorphism $X \mapsto X/K$ μ_j induce Gaussian distributions on the factor group X/K.

Lemma 7.1. *Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. The sum $\xi_1+\xi_2$ and the difference $\xi_1-\xi_2$ are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation*

$$\text{(i)}\quad \hat{\mu}_1(u+v)\hat{\mu}_2(u-v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(-v), \quad u, v \in Y.$$

Proof. We recall that if ξ is a random variable with values in a group X and distribution μ, then the characteristic function of μ is the expectation $\hat{\mu}(y) = \mathbf{E}[(\xi, y)]$. If ξ_1 and

ξ_2 are random variables taking values in X, then ξ_1 and ξ_2 are independent if and only if the equality

$$\mathbf{E}[(\xi_1, u)(\xi_2, v)] = \mathbf{E}[(\xi_1, u)]\mathbf{E}[(\xi_2, v)] \tag{7.1}$$

is fulfilled for all $u, v \in Y$. It follows from (7.1) that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent if and only if

$$\mathbf{E}[(\xi_1 + \xi_2, u)(\xi_1 - \xi_2, v)] = \mathbf{E}[(\xi_1 + \xi_2, u)]\mathbf{E}[(\xi_1 - \xi_2, v)], \quad u, v \in Y. \tag{7.2}$$

Taking into account that the random variables ξ_1 and ξ_2 are independent, we transform the left-hand side of equality (7.2) as follows:

$$\begin{aligned}\mathbf{E}[(\xi_1 + \xi_2, u)(\xi_1 - \xi_2, v)] &= \mathbf{E}[(\xi_1, u + v)(\xi_2, u - v)] \\ &= \mathbf{E}[(\xi_1, u + v)]\mathbf{E}[(\xi_2, u - v)] \\ &= \hat{\mu}_1(u + v)\hat{\mu}_2(u - v), \quad u, v \in Y.\end{aligned}$$

Analogously, we transform the right-hand side of equality (7.2)

$$\begin{aligned}\mathbf{E}[(\xi_1 + \xi_2, u)]\mathbf{E}[(\xi_1 - \xi_2, v)] &= \mathbf{E}[(\xi_1, u)(\xi_2, u)]\mathbf{E}[(\xi_1, v)(\xi_2, -v)] \\ &= \mathbf{E}[(\xi_1, u)]\mathbf{E}[(\xi_2, u)]\mathbf{E}[(\xi_1, v)]\mathbf{E}[(\xi_2, -v)] \\ &= \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(-v), \quad u, v \in Y.\end{aligned}$$ □

Equation (i) is called the *Kac–Bernstein functional equation*. In particular, it follows from (i) that an arbitrary Gaussian distribution $\gamma \in \Gamma(X)$ has the property: if ξ_1 and ξ_2 are independent identically distributed random variables with values in the group X and distribution γ, then $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Some idempotent distributions on the group X have the same property. Denote by $I_B(X)$ the set of such idempotent distributions. In other words, $m_K \in I_B(X)$ if ξ_1 and ξ_2 are independent identically distributed random variables with values in the group X and distribution m_K, then $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. To describe the set $I_B(X)$ we need the following lemma.

Lemma 7.2. *Let n be a natural number, and G be a closed subgroup of a group X. Then the following statements are equivalent:*

(i) $\overline{G^{(n)}} = G$;

(ii) *if* $ny \in A(Y, G)$, *then* $y \in A(Y, G)$.

Proof. (i) $\Longrightarrow$ (ii). By Theorem 1.9.2 $G^* \cong Y/A(Y, G)$. It is obvious that (ii) is equivalent to the fact that $(Y/A(Y, G))_{(n)} = \{0\}$. Therefore the equivalence of (i) and (ii) follows from Theorem 1.9.5. □

7.3 Corwin groups. A group X is called a *Corwin group* if $X^{(2)} = X$. We will give some examples.

The groups $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Q}$, $\mathbb{Z}(2k - 1)$, $\mathbb{Z}(p^\infty)$, $\Sigma_{\boldsymbol{a}}$, Δ_p, for $p \neq 2$ are Corwin groups.

The groups $\mathbb{Z}$, $\mathbb{Z}(2k)$, Δ_2 are not Corwin groups.

Proposition 7.4. *Let K be a compact subgroup of a group X. Then the following statements are equivalent:*

(i) *K is a Corwin group;*

(ii) *if $2y \in A(Y, K)$, then $y \in A(Y, K)$;*

(iii) *$m_K \in I_B(X)$.*

Proof. The equivalence of (i) and (ii) follows from Lemma 7.2 because $\overline{K^{(2)}} = K^{(2)}$.

(ii) $\Longrightarrow$ (iii). As has been noted in Lemma 7.1, we should verify that the characteristic function $\widehat{m}_K(y)$ satisfies equation 7.1 (i) which takes the form

$$\widehat{m}_K(u + v)\widehat{m}_K(u - v) = \widehat{m}_K^2(u)\widehat{m}_K^2(v), \quad u, v \in Y. \tag{7.3}$$

We use representation 2.16 (i) of the characteristic function $\widehat{m}_K(y)$. If $u, v \in A(Y, K)$, then $u \pm v \in A(Y, K)$ and both sides of equation (7.3) are equal to 1. If either $u \in A(Y, K)$, $v \notin A(Y, K)$ or $v \in A(Y, K)$, $u \notin A(Y, K)$, then $u + v \notin A(Y, K)$ and both sides of equation (7.3) are equal to zero. If $u, v \notin A(Y, K)$, then the right-hand side of equation (7.3) is equal to zero. If the left-hand side of equation (7.3) is not equal to zero, we have $u \pm v \in A(Y, K)$. This implies that $2u \in A(Y, K)$, and hence by (ii) $u \in A(Y, K)$, that contradicts the condition. Hence the left-hand side of equation (7.3) is also equal to zero.

(iii) $\Longrightarrow$ (ii). By Lemma 7.1 the characteristic function $\widehat{m}_K(y)$ satisfies equation (7.3). Let $2y \in A(Y, K)$. Substitute $u = v = y$ in (7.3). Then the left-hand side of equation (7.3) is equal to 1. Hence the right-hand side of equation (7.3) is equal to 1. Taking into account 2.16 (i), this means that $y \in A(Y, K)$. □

Our aim is to describe all groups X which have the following property: if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_1 and μ_2, then the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that $\mu_j \in \Gamma(X) * I(X)$. This inclusion means that the distributions μ_j are invariant with respect to a compact subgroup K of the group X and under the natural homomorphism $X \mapsto X/K$ induce Gaussian distributions on the factor group X/K.

To prove the main theorem of this section we need the following lemmas.

Lemma 7.5. *Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then distributions μ_j can be replaced by their shifts μ_j' in such a manner that $\sigma(\mu_j') \subset M$, $j = 1, 2$, where M is a subgroup of X such that M is topologically isomorphic to a group of the form $\mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group.*

Proof. Taking into account Theorem 1.11.1, we can assume without loss of generality that $X = \mathbb{R}^m \times G$, $Y = \mathbb{R}^m \times H$, where $m \geq 0$, $H \cong G^*$ and each of the groups G and H contains a compact open subgroup. Denote by L a compact open subgroup of H. Put

$$N_1 = \{y \in Y : \hat{\mu}_1(y) \neq 0\}, \quad N_2 = \{y \in Y : \hat{\mu}_2(y) \neq 0\}, \quad N = N_1 \cap N_2.$$

By Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). It follows from 7.1 (i) that N is a subgroup of Y. Obviously, N is an open subgroup.

Consider the intersection $B = N \cap L$. Since every open subgroup is closed, B is a compact open subgroup of H. Putting $u = v = y$ and then $u = -v = y$ into equation 7.1 (i) and taking into account 2.7 (d) we obtain

$$\hat{\mu}_1(2y) = \hat{\mu}_1^2(y)|\hat{\mu}_2(y)|^2, \quad \hat{\mu}_2(2y) = |\hat{\mu}_1(y)|^2\hat{\mu}_2^2(y), \qquad y \in Y. \tag{7.4}$$

We deduce from (7.4) that for every natural n the functions $\hat{\mu}_j(y)$ satisfy the equation

$$|\hat{\mu}_j(2^n y)| = |\hat{\mu}_1(y)\hat{\mu}_2(y)|^{2^{2n-1}}, \quad j = 1, 2, \ y \in Y. \tag{7.5}$$

For $n = 1$ this yields that

$$|\hat{\mu}_1(y)| = |\hat{\mu}_2(y)|, \quad y \in \overline{Y^{(2)}}. \tag{7.6}$$

Let $y \in B$. As B is compact, there exists a convergent subsequence $2^{m_k} y \to y_0 \in B$. We obtain from (7.5) that

$$|\hat{\mu}_j(y_0)| = \lim_{k\to\infty} |\hat{\mu}_j(2^{m_k} y)| = \lim_{k\to\infty} |\hat{\mu}_1(y)\hat{\mu}_2(y)|^{2^{2m_k-1}}, \ j = 1, 2, \ y \in Y. \tag{7.7}$$

If $|\hat{\mu}_1(y)\hat{\mu}_2(y)| < 1$, then the limit on the right-hand side of (7.7) is equal to zero, contrary to the fact that $y_0 \in B \subset N$. Hence $|\hat{\mu}_1(y)| = |\hat{\mu}_2(y)| = 1$ for all $y \in B$.

Taking into consideration 2.7 (e), we can replace the distributions μ_j by their shifts μ'_j in such a way that $\hat{\mu}'_j(y) = 1$ for all $y \in B$. It is obvious that the characteristic functions $\hat{\mu}'_j(y)$ also satisfy equation 7.1 (i). Applying Proposition 2.13 we get $\sigma(\mu'_j) \subset A(X, B)$. It follows from Theorems 1.7.1 and 1.9.2 that $A(X, B) \cong (Y/B)^* = ((\mathbb{R}^m \times H)/B)^* \cong \mathbb{R}^m \times (H/B)^*$. Put $F = (H/B)^*$. Since B is an open subgroup of H, the factor group H/B is discrete. Hence by Theorem 1.6.1, F is a compact group.

Thus we have reduced the proof of the lemma to the case when $X = \mathbb{R}^m \times F$, $Y = \mathbb{R}^m \times D$, where F is a compact group, $D \cong F^*$, and μ_j are distributions on X with characteristic functions satisfying equation 7.1 (i). By Theorem 1.6.1, D is a discrete group. Let D_2 be the 2-component of the group D, i.e., the subgroup of D consisting of all elements $y \in D$ such that the order of y is a power of the number 2. Let $y \in D_2$. Then $2^n y = 0$ for some natural n. We deduce from (7.5) that $|\hat{\mu}_1(y)| = |\hat{\mu}_2(y)| = 1$ for all $y \in D_2$. Applying 2.7 (e) we can replace the distributions μ_j by their shifts μ'_j in such a way that $\hat{\mu}'_j(y) = 1$ for all $y \in D_2$. By Proposition 2.13, $\sigma(\mu'_j) \subset A(X, D_2)$. Put $M = A(X, D_2)$. It is obvious that $M \cong \mathbb{R}^m \times K$, where K is a compact group. It follows from Theorems 1.9.1 and 1.9.2 that $M^* \cong Y/D_2$. It is clear that the factor group Y/D_2 contains no elements of order 2. By Theorem 1.9.5, $\overline{M^{(2)}} = M$. We conclude from $\overline{M^{(2)}} = M^{(2)}$ that M is a Corwin group. Hence K is also a Corwin group. Returning to the original distributions μ_j, we obtain the required statement. □

Lemma 7.6. *Let n be a natural number. The following statements are equivalent:*

(i) *for any compact subgroup G of a group X satisfying the condition $G^{(n)} = G$, the equality $(G^*)^{(n)} = G^*$ is fulfilled;*

(ii) *the connected component of zero c_X of a group X has the property $\{x \in c_X : nx = 0\} = \{0\}$.*

Proof. (i) $\Longrightarrow$ (ii). By Theorem 1.11.2, $c_X \cong \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact connected group. By Theorem 1.9.6, for every natural number l the equality $K^{(l)} = K$ is fulfilled. Then it follows from (i) that $(K^*)^{(n)} = K^*$. Hence $(c_X^*)^{(n)} = c_X^*$. By Theorem 1.9.5 the last equality implies (ii).

(ii) $\Longrightarrow$ (i). Let G be a compact subgroup of X such that $G^{(n)} = G$. Set $H = G^*$. By Theorem 1.6.1, H is a discrete group. We will verify that $H^{(n)} = H$. Consider the subgroup $\widetilde{H}$ of the group H consisting of all infinitely divisible by n elements, i.e.,

$$\widetilde{H} = \bigcap_{l=1}^{\infty} H^{(n^l)}.$$

First we will verify that the factor group $L = H/\widetilde{H}$ contains no nonzero elements that are infinitely divisible by n, and L contains no elements of finite order. Let $[h_0] \in L$ be an infinitely divisible by n element. Then the equation

$$n^l[t] = [h_0] \tag{7.8}$$

has a solution in L for every natural l. Equality (7.8) is equivalent to the fact that $n^l t - h_0 \in \widetilde{H}$, i.e., $n^l t - h_0 = n^l y$ for some $y \in H$. It follows from this that $h_0 = n^l(t - y)$, i.e., $h_0 \in \widetilde{H}$. Hence $[h_0] = 0$. We have proved that L contains no nonzero elements infinitely divisible by n. Assume now that $[h_0] \in L$ is an element of finite order p. We can suppose without loss of generality that p is a prime number. If n is not divided by p, then $[h_0]$ is an element infinitely divisible by n. As has been proved above, $[h_0] = 0$. Therefore we can assume that p is a factor of n. We have $p[h_0] = 0$, i.e., $ph_0 \in \widetilde{H}$. So for every natural l there exists an element $z \in H$ such that $ph_0 = n^{l+1}z$. We conclude from this that

$$p\left(h_0 - \frac{n^{l+1}}{p}z\right) = 0. \tag{7.9}$$

Taking into account that $G^{(n)} = G$ we have $\{h \in H : nh = 0\} = \{0\}$ by Theorem 1.9.5. Since p is a factor of n,

$$\{h \in H : ph = 0\} = \{0\}. \tag{7.10}$$

We obtain from (7.9) and (7.10) that $h_0 = n^l\left(\frac{nz}{p}\right)$, i.e., $h_0 \in \widetilde{H}$, and hence $[h_0] = 0$.

Thus L is a discrete torsion-free group. By Theorems 1.6.1 and 1.6.2 the group L^* is compact and connected. By Theorem 1.9.2, $L^* \cong A(G, \widetilde{H})$. Hence $A(G, \widetilde{H}) \subset c_X$, and (ii) implies that

$$\{x \in A(G, \widetilde{H}) : nx = 0\} = \{0\}.$$

By Theorem 1.9.5 it follows from this that $L^{(n)} = L$, i.e., the group L consists of elements infinitely divisible by n. Hence $L = \{0\}$. Thus $H = \widetilde{H}$, and hence $H^{(n)} = H$. □

Lemma 7.7. *The following statements are equivalent:*

(i) *for any compact Corwin subgroup K of a group X the factor group X/K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$;*

(ii) *if K is a compact Corwin subgroup K of a group X, then K^* is a Corwin group.*

Proof. (i) $\Rightarrow$ (ii). Let K be a compact Corwin subgroup of X. Put $L = K^*$. By Theorem 1.6.1, L is a discrete group. We will verify that $L^{(2)} = L$. It follows from Theorem 1.9.5 that L contains no elements of order 2. This implies that all elements of finite order of the group L have odd order. Hence they belong to $L^{(2)}$. We will verify that every element of infinite order of L belongs to $L^{(2)}$. Let $h_0 \notin L^{(2)}$. Denote by $M = \{h \in L : h = nh_0, n \in \mathbb{Z}\}$ the subgroup of L generated by the element h_0. Then $M \cong \mathbb{Z}$. Let $h \in L$ and let $2h \in M$. If $2h = (2m-1)h_0$, then $h_0 = 2(mh_0 - h) \in L^{(2)}$, which contradicts the choice of h_0. Hence if $2h \in M$, then $2h = 2mh_0$. Inasmuch as L contains no elements of order 2, $h = mh_0$. Thus the subgroup M has the property that if $2h \in M$, then $h \in M$. Taking into account Theorem 1.9.1, it follows from Lemma 7.2 that the annihilator $A(K, M)$ is a Corwin group. Obviously, $A(K, M)$ is compact. Applying Theorems 1.9.1 and 1.9.2 we get $(K/A(K, M))^* \cong M \cong \mathbb{Z}$. By Theorem 1.17.2 the subgroup M is a direct factor of L. Hence the factor group $K/A(K, M)$ contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$.

Note now that if a group X satisfies condition (i), then every closed subgroup of X also satisfies condition (i). We obtain a contradiction with (i). Thus (ii) is true.

(ii) $\Rightarrow$ (i). First we verify that if G is a compact subgroup of X, then the factor group X/G also satisfies condition (ii). Let $p\colon X \mapsto X/G$ be the natural homomorphism, K be a compact Corwin subgroup of X/G. Put $\widetilde{K} = p^{-1}(K)$. Then $K \cong \widetilde{K}/G$. In view of compactness of the groups K and G the group $\widetilde{K}$ is also compact. Obviously, $\widetilde{K}$ is a Corwin group. We deduce from (ii) that $\widetilde{K}^*$ is a Corwin group. By Theorem 1.9.2, $K^* \cong A(\widetilde{K}^*, G)$. Let $y \in A(\widetilde{K}^*, G)$. Then $y = 2y'$, $y' \in \widetilde{K}^*$. Since G is a Corwin group, by Lemma 7.2, $y' \in A(\widetilde{K}^*, G)$, i.e., K^* is a Corwin group.

If condition (i) is not satisfied, then there exists a compact Corwin subgroup K_0 such that the factor group X/K_0 contains a subgroup F topologically isomorphic to the circle group $\mathbb{T}$. Since $\mathbb{T}$ is a compact Corwin group, it follows from what has been said above that F^* is also a Corwin group. But it is obviously false. Hence condition (i) is fulfilled. □

Lemma 7.8. *Let $X = \mathbb{T}$. Then there exist independent random variables ξ_1 and ξ_2 with values in $\mathbb{T}$ and distributions μ_1 and μ_2 such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, $\hat{\mu}_j(y) \neq 0$ for all $y \in Y$ and $\mu_j \notin \Gamma(\mathbb{T})$, $j = 1, 2$.*

Proof. We have $Y \cong \mathbb{Z}$. We will suppose without loss of generality that $Y = \mathbb{Z}$. Consider on the group $\mathbb{Z}$ the functions

$$g_1(n) = \begin{cases} \exp\{-an^2\} & \text{if } n \in \mathbb{Z}^{(2)}, \\ b\exp\{-an^2\} & \text{if } n \notin \mathbb{Z}^{(2)}, \end{cases}$$
$$g_2(n) = \begin{cases} \exp\{-an^2\} & \text{if } n \in \mathbb{Z}^{(2)}, \\ b^{-1}\exp\{-an^2\} & \text{if } n \notin \mathbb{Z}^{(2)}. \end{cases} \tag{7.11}$$

Choose real numbers a and b in such a way that

$$\sum_{n\in\mathbb{Z}, n\neq 0} g_j(n) < 1, \quad j = 1, 2.$$

It follows from this that

$$\rho_j(t) = \sum_{n=-\infty}^{\infty} g_j(n)\exp\{-int\} > 0, \quad t \in \mathbb{R},\ j = 1, 2.$$

It is also obvious that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_j(t)\,dt = 1, \quad j = 1, 2.$$

Let μ_j be the distribution on the circle group $\mathbb{T}$ with density $r_j(e^{it}) = \rho_j(t)$ with respect to $m_{\mathbb{T}}$. Then $\hat{\mu}_j(n) = g_j(n)$. Let ξ_1 and ξ_2 be independent random variables with values in $\mathbb{T}$ and distributions μ_1 and μ_2. It is obvious that $\mu_j \notin \Gamma(\mathbb{T})$ and $\hat{\mu}_j(n) \neq 0$ for all $n \in \mathbb{Z}$. We will verify that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Thus the lemma will be proved.

By Lemma 7.1 it suffices to show that the characteristic functions $\hat{\mu}_j(n)$ satisfy equation 7.1 (i). By considering separately the cases $m, n \in \mathbb{Z}^{(2)}$; $m \in \mathbb{Z}^{(2)}, n \notin \mathbb{Z}^{(2)}$; $n \in \mathbb{Z}^{(2)}, m \notin \mathbb{Z}^{(2)}$; $m, n \notin \mathbb{Z}^{(2)}$ we are convinced of this fact. □

As will be proved in Section 8 (see Corollary 8.6), if ξ_1 and ξ_2 are independent random variables with values in the circle group $\mathbb{T}$ and distributions μ_1 and μ_2 with non-vanishing characteristic functions, then the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that the distributions μ_j can be replaced by their shifts μ'_j in such a way that the characteristic functions of the distributions μ'_j are defined by (7.11).

Lemma 7.9. *Let a group X contain no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X)$ and $\mu_1 = \mu_2 * E_x$, $x \in X$.*

Proof. By Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). Replacing in equation 7.1 (i) v by $-v$, we obtain

$$\hat{\mu}_1(u-v)\hat{\mu}_2(u+v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(-v)\hat{\mu}_2(v), \quad u, v \in Y. \tag{7.12}$$

Put $\lambda = \mu_1 * \mu_2$. We conclude from 2.7 (b) that $\hat{\lambda}(y) = \hat{\mu}_1(y)\hat{\mu}_2(y)$. Multiplying equations 7.1 (i) and (7.12) and taking into account 2.7 (d), we get

$$\hat{\lambda}(u+v)\hat{\lambda}(u-v) = \hat{\lambda}^2(u)|\hat{\lambda}(v)|^2, \quad u, v \in Y. \tag{7.13}$$

Let $\nu = \lambda * \bar{\lambda}$. Applying 2.7 (c) and 2.7 (d) we obtain $\hat{\nu}(y) = |\lambda(y)|^2 > 0$ and find from (7.13) that

$$\hat{\nu}(u+v)\hat{\nu}(u-v) = \hat{\nu}^2(u)\hat{\nu}^2(v), \quad u, v \in Y. \tag{7.14}$$

Put $\varphi(y) = -\ln\hat{\nu}(y)$. It follows from (7.14) that the characteristic function $\hat{\nu}(y)$ is represented in the form

$$\hat{\nu}(y) = \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $\varphi(y)$ is a continuous nonnegative function on Y satisfying equation 2.16 (ii). So, $\nu \in \Gamma(X)$. The distributions μ_j are factors of ν. Since the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Theorem 4.6, $\mu_j \in \Gamma(X)$. Hence

$$\hat{\mu}_j(y) = (x_j, y)\exp\{-\varphi_j(y)\},$$

where $x_j \in X$, and $\varphi_j(y)$ is a continuous nonnegative function on the group Y satisfying equation 2.16 (ii). Putting the expression for $\hat{\mu}_j(y)$ into equation 7.1 (i) we get

$$\varphi_1(u+v) + \varphi_2(u-v) = \varphi_1(u) + \varphi_2(u) + \varphi_1(v) + \varphi_2(-v), \quad u, v \in Y. \tag{7.15}$$

We use Remark 3.3 and pass in equation (7.15) to the corresponding 2-additive functions $\psi_j(u, v)$. We have

$$\psi_1(u, v) = \psi_2(u, v), \quad u, v \in Y,$$

and hence $\varphi_1(y) = \varphi_2(y)$, $y \in Y$. Thus $\mu_1 = \mu_2 * E_x$, $x \in X$. □

It should be noted that the statement of Lemma 7.9 does not remain valid if the group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$. This results from Lemma 7.8.

Now we prove the main theorem of this section.

Theorem 7.10. *The following statements are valid:*

(I) *Assume that the connected component of zero of a group X contains no elements of order* 2*. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Then $\mu_j \in \Gamma(X) * I_B(X)$ and $\mu_1 = \mu_2 * E_x$, $x \in X$.*

(II) *If the connected component of zero of a group X contains elements of order 2, then there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\lambda_j \notin \Gamma(X) * I(X)$ such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent.*

Proof. (I). By Lemma 7.5, we can assume that $X = \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Applying Lemma 7.6 for $n = 2$ we obtain that $(K^*)^{(2)} = K^*$, and hence $Y^{(2)} = Y$. It is also obvious that $X^{(2)} = X$. Since $\overline{X^{(2)}} = X^{(2)}$, it follows from 1.13 (b) and 1.13 (d) that f_2 is a topological automorphism of the groups X and Y, i.e., X and Y are groups with unique division by 2.

By Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). Hence they also satisfy equations (7.5) and (7.6). Since $Y^{(2)} = Y$, it follows from (7.6) that $|\hat{\mu}_1(y)| = |\hat{\mu}_2(y)|$, $y \in Y$. Therefore

$$\{y \in Y : \hat{\mu}_1(y) \neq 0\} = \{y \in Y : \hat{\mu}_2(y) \neq 0\} = N.$$

Equation 7.1 (i) implies that N is a subgroup of Y. It is obvious that N is an open subgroup. By equation (7.5) for $n = 1$ we get that if $2y \in N$, then $y \in N$. Applying Lemma 7.2 for $n = 2$ we obtain that $F = A(X, N)$ is a Corwin group. Since N is an open subgroup, by Theorem 1.9.4, F is a compact subgroup. It follows from Theorems 1.9.1 and 1.9.2 that $(X/F)^* \cong N$. It is also easily seen that X/F and N are groups with unique division by 2.

Consider the restrictions of the characteristic functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ to the subgroup N. In view of Corollary 2.11 and Lemma 7.1 these restrictions are the characteristic functions of some independent random variables ζ_1 and ζ_2 taking values in the factor group X/F and having the property that $\zeta_1 + \zeta_2$ and $\zeta_1 - \zeta_2$ are independent. Since X/F is a group with unique division by 2, X/F contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Applying Lemma 7.9 to the factor group X/F we get that the restrictions of the characteristic functions $\hat{\mu}_j(y)$ to the subgroup N have the representations

$$\hat{\mu}_j(y) = (x_j, y) \exp\{-\varphi(y)\}, \quad y \in N, \tag{7.16}$$

where $x_j \in X$, and $\varphi(y)$ is a continuous nonnegative function on N satisfying equation 2.16 (ii). By Lemma 3.18 we can extend the function $\varphi(y)$ from the subgroup N to Y in such a manner that its properties are preserved. Let $\tilde{\varphi}(y)$ denote the extended function.

Let γ_j be Gaussian distributions on the group X with the characteristic functions

$$\hat{\gamma}_j(y) = (x_j, y) \exp\{-\tilde{\varphi}(y)\}, \quad y \in Y, \ j = 1, 2. \tag{7.17}$$

By Theorem 1.9.1, $N = A(Y, F)$. Then it follows from 2.14 (i) that the characteristic function of the Haar distribution m_F of the subgroup F is of the form

$$\widehat{m}_F(y) = \begin{cases} 1 & \text{if } y \in N, \\ 0 & \text{if } y \notin N. \end{cases} \tag{7.18}$$

We deduce from (7.16)–(7.18) that $\hat{\mu}_j(y) = \hat{\gamma}_j(y)\widehat{m}_F(y)$. Hence applying 2.7 (b) and 2.7 (c), we get $\mu_j = \gamma_j * m_F$. Since $\gamma_1 = \gamma_2 * E_x$, we have $\mu_1 = \mu_2 * E_x$. Statement (I) is proved.

Let us prove (II). Assume that the connected component of zero of the group X contains elements of order 2. Applying Lemma 7.6 for $n = 2$ we get that there exists a compact Corwin subgroup G of the group X such that G^* is not a Corwin group. Then by Lemma 7.7 there exists a compact subgroup K of the group X such that the factor group X/K contains a subgroup F topologically isomorphic to the circle group $\mathbb{T}$. For this reason the distributions μ_j on the circle group $\mathbb{T}$ constructed in Lemma 7.8 can be considered as distributions on the factor group X/K. We will retain for them the notation μ_j. Taking into account that by Theorem 1.9.2, $(X/K)^* \cong A(Y, K)$, we can assume that the characteristic functions $\hat{\mu}_j(y)$ are defined on $A(Y, K)$. Consider on the group Y the functions

$$h_j(y) = \begin{cases} \hat{\mu}_j(y) & \text{if } y \in A(Y, K), \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

Since $A(Y, K)$ is a subgroup and $\hat{\mu}_j(y)$ are positive definite functions, by Proposition 2.12, $h_j(y)$ are also positive definite functions. Since K is a compact group, by Theorem 1.9.4 the annihilator $A(Y, K)$ is an open subgroup. Hence the functions $h_j(y)$ are continuous. By the Bochner theorem there exist distributions $\lambda_j \in \mathrm{M}^1(X)$ such that $\hat{\lambda}_j(y) = h_j(y)$. Let ξ_j be independent random variables with values in the group X and distributions λ_j.

It is easy to verify that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Indeed by Lemma 7.1 it suffices to show that the characteristic functions $\hat{\lambda}_j(y)$ satisfy equation 7.1 (i). Let $u, v \in A(Y, K)$. Then it is obvious that 7.1 (i) holds, because the functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). If either $u \in A(Y, K)$, $v \notin A(Y, K)$ or $v \in A(Y, K)$, $u \notin A(Y, K)$, then both sides of equation 7.1 (i) are equal to zero. If $u, v \notin A(Y, K)$, then the right-hand side of equation 7.1 (i) is equal to zero. If the left-hand side of equation 7.1 (i) is not equal to zero, then $u \pm v \in A(Y, K)$. This implies that $2u \in A(Y, K)$. Since K is a compact Corwin group, applying Lemma 7.2 for $n = 2$ we get that $u \in A(Y, K)$. This contradicts the assumption. Thus the left-hand side of 7.1 (i) is also equal to zero. We have proved that the characteristic functions $\hat{\lambda}_j(y)$ satisfy equation 7.1 (i). Since $\mu_j \notin \Gamma(X) * I(X)$, it is obvious that $\lambda_j \notin \Gamma(X) * I(X)$. Statement (II) is also proved. □

Remark 7.11. We note that in the proof of statement (I) in Theorem 7.10 we did not use the Kac–Bernstein theorem. Hence this theorem follows from Theorem 7.10, because the only compact Corwin subgroup K of the group $\mathbb{R}$ is $K = \{0\}$.

Remark 7.12. The proof of statement (I) in Theorem 7.10 is based on Lemma 7.9. To prove Lemma 7.9 we used Theorem 4.6 (the group analogue of the Cramér theorem). In turn the proof of Theorem 4.6 is based on the Cramér theorem for the group $\mathbb{R}^m$ (see Theorem 2.18). All known proofs of Theorem 2.18 use functions of a complex variable, in particular the theory of entire functions.

It is easy to see that to prove statement (I) in Theorem 7.10 it suffices to prove the following statement which is weaker than Lemma 7.9.

Let X be a group with unique division by 2. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. If $\xi_1+\xi_2$ and $\xi_1-\xi_2$ are independent, then $\mu_j \in \Gamma(X)$ and $\mu_1=\mu_2 * E_x$, $x\in X$.

Below we give two proofs of this statement without using Theorem 4.6, and hence without using the theory of entire functions. We need the following lemma.

Lemma 7.13. *Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Let $p\colon X \mapsto X/X_{(2)}$ be the natural homomorphism. If $\xi_1+\xi_2$ and $\xi_1-\xi_2$ are independent, then $p(\mu_j)\in\Gamma(X/X_{(2)})$ and $p(\mu_1)=p(\mu_2)*E_{[x]}$, $[x]\in X/X_{(2)}$.*

Proof. By Lemma 7.1 the characteristic functions $\hat\mu_j(y)$ satisfy equation 7.1 (i). Hence they satisfy equation (7.6). In view of 2.7 (d) we have

$$|\hat\mu_2(-v)| = |\hat\mu_2(v)|, \quad v\in Y. \tag{7.19}$$

It follows from equation 7.1 (i) and (7.19) that the functions $|\hat\mu_j(y)|$ satisfy the equation

$$|\hat\mu_1(u+v)||\hat\mu_2(u-v)| = |\hat\mu_1(u)||\hat\mu_1(v)||\hat\mu_2(u)||\hat\mu_2(v)|, \quad u,v\in Y. \tag{7.20}$$

Using (7.6) and (7.20) we obtain

$$|\hat\mu_j(u+v)||\hat\mu_j(u-v)| = |\hat\mu_j(u)|^2|\hat\mu_j(v)|^2, \quad u,v\in \overline{Y^{(2)}},\ j=1,2.$$

This implies that $\varphi(y) = -\ln|\hat\mu_j(y)|$ is a continuous nonnegative function on the group $\overline{Y^{(2)}}$ satisfying equation 2.16 (ii).

Let us check that the functions

$$l_j(y) = \hat\mu_j(y)/|\hat\mu_j(y)|$$

are characters of the group $\overline{Y^{(2)}}$. Note that

$$|l_j(y)|=1, \quad l_j(-y)=\overline{l_j(y)}, \quad l_j(0)=1, \quad j=1,2,\ y\in Y. \tag{7.21}$$

It is obvious that the functions $l_j(y)$ satisfy equation 7.1 (i) which takes the form

$$l_1(u+v)l_2(u-v) = l_1(u)l_1(v)l_2(u)l_2(-v), \quad u,v\in Y. \tag{7.22}$$

Putting $u=v=y$ and then $u=-v=y$ into equation (7.22) we get

$$l_1(2y)=l_1^2(y), \quad l_2(2y)=l_2^2(y), \quad y\in Y. \tag{7.23}$$

Changing places of u and v in (7.22) we infer that

$$l_1(u+v)l_2(-(u-v)) = l_1(u)l_1(v)l_2(-u)l_2(v), \quad u,v\in Y.$$

Multiplying this equation and equation (7.22) and taking into account (7.21), we obtain

$$l_1^2(u+v) = l_1^2(u) l_1^2(v), \quad u, v \in Y. \tag{7.24}$$

In view of (7.23) equality (7.24) implies that

$$l_1(u+v) = l_1(u) l_1(v), \quad u, v \in \overline{Y^{(2)}}.$$

A similar argument proves that the function $l_2(y)$ is also a character of the group $\overline{Y^{(2)}}$. By Theorem 1.9.2 the characters $l_j(y)$ can be represented in the form $l_j(y) = (x_j, y)$, $x_j \in X$, $y \in \overline{Y^{(2)}}$. Thus we have proved that the characteristic functions $\hat{\mu}_j(y)$ on the subgroup $\overline{Y^{(2)}}$ can be represented in the form

$$\hat{\mu}_j(y) = (x_j, y) \exp\{-\varphi(y)\}, \quad y \in \overline{Y^{(2)}}. \tag{7.25}$$

We note that by Theorem 1.9.5, $\overline{Y^{(2)}} = A(Y, X_{(2)})$. Let $p\colon X \mapsto X/X_{(2)}$ be the natural homomorphism. Since by Corollary 2.11 the restriction of the characteristic function $\hat{\mu}_j(y)$ to the subgroup $\overline{Y^{(2)}}$ is the characteristic function of the distribution $p(\mu_j) \in X/X_{(2)}$, the lemma is proved. □

Corollary 7.14. *Let a group X satisfy the condition $X_{(2)} = \{0\}$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X)$ and $\mu_1 = \mu_2 * E_x$, $x \in X$.*

Obviously, Corollary 7.14 yields the statement presented in Remark 7.12.

Remark 7.15. It is easy to see that we can conclude from the proof of Lemma 7.13 the following statement. Let Y be an arbitrary Abelian group. Let the functions $f_j(y)$ on the group Y satisfy the equation

$$f_1(u+v) f_2(u-v) = f_1(u) f_2(u) f_1(v) f_2(-v), \quad u, v \in Y, \tag{7.26}$$

and the conditions $f_j(-y) = \overline{f_j(y)}$, $y \in Y$, $f_j(0) = 1$, $j = 1, 2$. Then on the subgroup $Y^{(2)}$ the representation

$$f_j(y) = l_j(y) \exp\{\varphi(y)\}, \quad y \in Y^{(2)}, \tag{7.27}$$

holds. Here $l_j(y)$ are functions on the subgroup $Y^{(2)}$ satisfying the equation

$$l_j(u+v) = l_j(u) l_j(v), \quad u, v \in Y^{(2)}, \tag{7.28}$$

and $\varphi(y)$ is a function on the subgroup $Y^{(2)}$ satisfying the equation 2.14 (ii).

If we assume additionally that Y is a locally compact group and the functions $f_j(y)$ are continuous, we can assert that on the subgroup $\overline{Y^{(2)}}$ the representation

$$f_j(y) = (x_j, y) \exp\{\varphi(y)\}, \quad y \in \overline{Y^{(2)}}, \tag{7.29}$$

holds. Here $x_j \in X = Y^*$, and the function $\varphi(y)$ is continuous and satisfies equation 2.14 (ii).

Below we give another proof of the statement presented in Remark 7.12. It is of independent interest. For groups with unique division by 2, denote by $x_0/2$ the solution of the equation $2x = x_0$.

7.16. If X is a group with unique division by 2, then $Y = Y^{(2)}$. Hence as has been noted in the proof of Lemma 7.13,

$$|\hat{\mu}_j(y)| = \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $\varphi(y)$ is a continuous nonnegative function on Y satisfying equation 2.16 (ii). We will prove that the functions $l_j(y) = \hat{\mu}_j(y)/|\hat{\mu}_j(y)|$ are characters of the group Y. In view of (7.6) it follows from (7.4) that

$$\hat{\mu}_j(y) = (\hat{\mu}_j(y/2^n))^{2^n} |\hat{\mu}_j(y/2^n)|^{2^{2n}-2^n}, \quad y \in Y, \ j = 1, 2. \tag{7.30}$$

Note that the equality $\varphi(y/2^n) = (1/2^{2n})\varphi(y)$ implies that

$$\lim_{n\to\infty} |\hat{\mu}_j(y/2^n)|^{2^{2n}-2^n} = |\hat{\mu}_j(y)|, \quad y \in Y, \ j = 1, 2. \tag{7.31}$$

Rewrite (7.30) in the form

$$\frac{\hat{\mu}_j(y)}{|\hat{\mu}_j(y/2^n)|^{2^{2n}-2^n}} = (\hat{\mu}_j(y/2^n)^{2^n}, \quad y \in Y, \ j = 1, 2.$$

Passing here to the limit as $n \to \infty$ and taking into account (7.31), we get

$$l_j(y) = \frac{\hat{\mu}_j(y)}{|\hat{\mu}_j(y)|} = \lim_{n\to\infty} (\hat{\mu}_j(y/2^n))^{2^n}, \quad y \in Y, \ j = 1, 2.$$

Obviously, the functions $l_j(y)$ are continuous. Since the functions $l_j(y)$ are limits of a sequence of positive definite functions, the functions $l_j(y)$ are also positive definite. By the Bochner theorem $l_j(y)$ are characteristic functions. It follows from $|l_j(y)| = 1$, $y \in Y$, and 2.7 (e) that $l_j(y) = (x_j, y)$, $x_j \in X$. Thus $\hat{\mu}_j(y) = (x_j, y)\exp\{-\varphi(y)\}$, $y \in Y$. This implies that $\mu_j \in \Gamma(X)$, $j = 1, 2$, and $\mu_1 = \mu_2 * E_x$, $x \in X$. The assertion is proved.

Remark 7.17. Let ξ_1 and ξ_2 be independent random variables with values in a group X and infinitely divisible distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X) * I_B(X)$ and $\mu_1 = \mu_2 * E_x$, $x \in X$. Indeed, in view of Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). Hence they also satisfy equation (7.4). We conclude from 2.15 (b) that the sets $N_j = \{y \in Y : \hat{\mu}_j(y) \neq 0\}$, $j = 1, 2$, are subgroups of Y, and (7.4) implies that $N_1 = N_2$.

The further reasoning is the same as in the proof of Theorem 7.10, but instead of Lemma 7.9 we use the following statement.

Let ξ_1 and ξ_2 be independent random variables with values in a group X and infinitely divisible distributions μ_1 and μ_2 with non-vanishing characteristic functions. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X)$ and $\mu_1 = \mu_2 * E_x$, $x \in X$.

The proof of this assertion is similar to the proof of Lemma 7.9, but instead of Theorem 4.6 one should use Remark 4.8.

8 Random variables with values in the group $\mathbb{R}\times\mathbb{T}$ and in the $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$

Let X be a second countable locally compact Abelian group, Y be its character group. According to Theorem 7.10, if the connected component of zero of the group X contains no elements of order 2 and ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_1 and μ_2 such that their sum and difference are independent, then μ_j are invariant with respect to some compact subgroup K of X and under the natural homomorphism $X \mapsto X/K$ induce Gaussian distributions on the factor group X/K. Assume that the connected component of zero of a group X contains elements of order 2. The following natural problem arises: to describe distributions of independent random variables ξ_j taking values in X and having independent sum and difference. In this section we solve this problem for the group $\mathbb{R}\times\mathbb{T}$ and for $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$.

8.1. Consider the group $X=\mathbb{R}\times\mathbb{T}$. It is convenient to assume that the real line $\mathbb{R}$, the circle group $\mathbb{T}$ and the multiplicative group of mth roots of unity $\mathbb{Z}(m)$ are embedded in the natural way into X. The character group of the group X is topologically isomorphic to the group $\mathbb{R}\times\mathbb{Z}$. To avoid introducing new notation we will suppose that $Y=\mathbb{R}\times\mathbb{Z}$. We will also assume that the real line $\mathbb{R}$ and the group of integers $\mathbb{Z}$ are also embedded in the natural way into Y. We denote by $x=(t,z)$, $t\in\mathbb{R}$, $z\in\mathbb{T}$, elements of the group X, and by $y=(s,n)$, $s\in\mathbb{R}$, $n\in\mathbb{Z}$, elements of the group Y.

To prove the main theorems of this section we need the following lemmas.

Lemma 8.2. *Assume that complex-valued functions $h_j(n)$ on the group $\mathbb{Z}$ satisfy the equation*

$$\text{(i)}\quad h_1(m+n)h_2(m-n)=h_1(m)h_1(n)h_2(m)h_2(-n),\quad m,n\in\mathbb{Z},$$

and the conditions

$$\text{(ii)}\quad h_1(n)h_2(n)\neq 0,\quad h_j(-n)=\overline{h_j(n)},\quad h_j(0)=1,\quad j=1,2,\ n\in\mathbb{Z}.$$

Then the functions $h_j(n)$ can be represented in the form

$$h_1(n)=\exp\{-\lambda n^2+in\theta_1+\kappa(1-(-1)^n)\},\quad n\in\mathbb{Z},$$
$$h_2(n)=\exp\{-\lambda n^2+in\theta_2-\kappa(1-(-1)^n)\},\quad n\in\mathbb{Z},$$

where λ, $\kappa\in\mathbb{R}$, $0\le\theta_j<2\pi$, $j=1,2$.

Proof. It follows from equation (i) and condition (ii) that the functions $|h_j(n)|$ satisfy the equation

$$|h_1(m+n)||h_2(m-n)|=|h_1(m)||h_1(n)||h_2(m)||h_2(n)|,\quad m,n\in\mathbb{Z}. \tag{8.1}$$

Put $f(n)=-\ln|h_1(n)|$, $g(n)=-\ln|h_2(n)|$. From (8.1) we find

$$f(m+n)+g(m-n)=P(m)+P(n),\quad m,n\in\mathbb{Z}, \tag{8.2}$$

where $P(m) = f(m) + g(m)$.

We employ the finite-difference method for solving equation (8.2). Let k be an arbitrary element of $\mathbb{Z}$. Substitute $m + k$ for m and $n + k$ for n in equation (8.2). Subtracting (8.2) from the resulting equation we obtain

$$\Delta_{2k} f(m+n) = \Delta_k P(m) + \Delta_k P(n), \quad k, m, n \in \mathbb{Z}. \tag{8.3}$$

Putting $m = 0$ in (8.3) and subtracting the resulting equation from (8.3) we obtain that

$$\Delta_{2k}\Delta_m f(n) = \Delta_k P(m) - \Delta_k P(0), \quad k, m, n \in \mathbb{Z}. \tag{8.4}$$

Let l be an arbitrary element of $\mathbb{Z}$. Substitute $n + l$ for n in equation (8.4) and subtract (8.4) from the resulting equation. We have

$$\Delta_{2k}\Delta_m\Delta_l f(n) = 0, \quad k, m, l, n \in \mathbb{Z}.$$

Putting here $m = l$ we get

$$\Delta_{2k}\Delta_l^2 f(n) = 0, \quad k, l, n \in \mathbb{Z}. \tag{8.5}$$

Put $l = k$ in (8.5) and rewrite the obtained equation in the form

$$f(n+4k) - 2f(n+3k) + 2f(n+k) - f(n) = 0, \quad k, n \in \mathbb{Z}.$$

Putting here $k = 1$ we have

$$f(n+4) - 2f(n+3) + 2f(n+1) - f(n) = 0, \quad n \in \mathbb{Z}. \tag{8.6}$$

We have arrived at a finite difference equation on $\mathbb{Z}$. The corresponding characteristic equation is of the form

$$q^{n+4} - 2q^{n+3} + 2q^{n+1} - q^n = 0.$$

This implies that the general solution of equation (8.6) can be written as follows

$$f(n) = a_1 + b_1 n + c_1 n^2 + d_1(-1)^n, \quad n \in \mathbb{Z},$$

where a_1, b_1, c_1, d_1 are arbitrary real constants (see, e.g., [53], Chapter V, § 4). Since $f(0) = 0$ and $f(n) = f(-n)$ for all $n \in \mathbb{Z}$, we obtain that $b_1 = 0$ and $d_1 = -a_1$. Thus $f(n) = c_1 n^2 + a_1(1 - (-1)^n)$.

Arguing as above we find that $g(n) = c_2 n^2 + a_2(1 - (-1)^n)$. Substituting the obtained expressions for functions $f(n)$ and $g(n)$ into (8.2) we find that $c_1 = c_2$, $a_2 = -a_1$. Put $c_1 = \lambda$, $a_1 = -\kappa$. Thus $f(n) = \lambda n^2 - \kappa(1 - (-1)^n)$, $g(n) = \lambda n^2 + \kappa(1 - (-1)^n)$, i.e.,

$$\begin{aligned} |h_1(n)| &= \exp\{-\lambda n^2 + \kappa(1 - (-1)^n)\}, \\ |h_2(n)| &= \exp\{-\lambda n^2 - \kappa(1 - (-1)^n)\}, \quad n \in \mathbb{Z}. \end{aligned} \tag{8.7}$$

Consider the functions

$$l_j(n) = h_j(n)/|h_j(n)|, \quad n \in \mathbb{Z}. \tag{8.8}$$

Then $|l_j(n)| = 1$, $l_j(-n) = \overline{l_j(n)}$, $l_j(0) = 1$, $j = 1, 2, n \in \mathbb{Z}$. We deduce from (i) that the functions $l_j(n)$ satisfy the equation

$$l_1(m+n)l_2(m-n) = l_1(m)l_1(n)l_2(m)l_2(-n), \quad m, n \in \mathbb{Z}. \tag{8.9}$$

Since $|l_j(1)| = 1$, we have $l_j(1) = e^{i\theta_j}$ for some $0 \le \theta_j < 2\pi$, $j = 1, 2$. In view of equation (8.9) it is easy to prove by induction that

$$l_j(n) = e^{in\theta_j}, \quad j = 1, 2, \ n \in \mathbb{Z}. \tag{8.10}$$

The assertion of the lemma follows from (8.7), (8.8) and (8.10). □

Lemma 8.3. *For a given $\kappa \in \mathbb{R}$ the functions*

$$a_1(s,n) = \exp\{\kappa(1-(-1)^n)\}, \quad a_2(s,n) = \exp\{-\kappa(1-(-1)^n)\}$$

on the group $\mathbb{R} \times \mathbb{Z}$ are the characteristic functions of the signed measures

$$\text{(i)} \quad \pi_1 = \frac{1}{2}(1+e^{2\kappa})E_{(0,1)} + \frac{1}{2}(1-e^{2\kappa})E_{(0,-1)},$$

and

$$\text{(ii)} \quad \pi_2 = \frac{1}{2}(1+e^{-2\kappa})E_{(0,1)} + \frac{1}{2}(1-e^{-2\kappa})E_{(0,-1)},$$

concentrated on the subgroup $\mathbb{Z}(2) \subset \mathbb{R} \times \mathbb{T}$. Moreover $\pi_1 * \pi_2 = E_{(0,1)}$.

Proof. Direct verification. □

8.4. Consider the group $X = \mathbb{R} \times \mathbb{T}$. Let $\tau\colon X \mapsto X$ be the homomorphism defined by

$$\tau(t,z) = (t, z^2), \quad (t,z) \in \mathbb{R} \times \mathbb{T}. \tag{8.11}$$

Then the adjoint homomorphism $\tilde{\tau}\colon Y \mapsto Y$ is of the form $\tilde{\tau}(s,n) = (s, 2n)$, $(s,n) \in \mathbb{R} \times \mathbb{Z}$. Let $\mu \in \mathrm{M}^1(X)$. We conclude from Proposition 2.10 that the characteristic function of the distribution $\tau(\mu)$ is of the form $\widehat{\tau(\mu)}(s,n) = \hat{\mu}(s,2n)$. Now we can prove the main theorem of this section.

Theorem 8.5. *Let ξ_1 and ξ_2 be independent random variables with values in the group $X = \mathbb{R} \times \mathbb{T}$ and distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then one of the following statements is true:*

(i) *$\mu_j = \gamma * \pi_j * E_{x_j}$, where $\gamma \in \Gamma(X)$, $x_j \in X$, and π_j are signed measures concentrated on the subgroup $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$;*

(ii) *$\mu_j = \gamma * m_{\mathbb{Z}(p)} * \pi_j * E_{x_j}$, where $\gamma \in \Gamma(X)$, $x_j \in X$, p is an odd number, and π_j are signed measures concentrated on the subgroup $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$;*

(iii) *either* $\tau(\mu_1) = \gamma * m_{\mathbb{T}} * E_{x_1}$ *and* $\mu_2 = \gamma * m_{\mathbb{T}} * E_{x_2}$ *or* $\mu_1 = \gamma * m_{\mathbb{T}} * E_{x_1}$ *and* $\tau(\mu_2) = \gamma * m_{\mathbb{T}} * E_{x_2}$, *where the homomorphism* $\tau\colon X \mapsto X$ *is defined by* (8.11), $\gamma \in \Gamma(\mathbb{R})$, $x_j \in \mathbb{R}$.

Proof. By Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). Hence they also satisfy equations (7.4), (7.6) and (7.20). Put

$$N_1 = \{y \in Y : \hat{\mu}_1(y) \neq 0\}, \quad N_2 = \{y \in Y : \hat{\mu}_2(y) \neq 0\}, \quad N = N_1 \cap N_2.$$

It follows from equation 7.1 (i) that N is an open subgroup of Y. It is easy to see that there are three possibilities for N: $N = \mathbb{R} \times \mathbb{Z}$, $N = \mathbb{R} \times \mathbb{Z}^{(p)}$, where $p \neq 1$, and $N = \mathbb{R}$.

1. $N = \mathbb{R} \times \mathbb{Z}$. First we will find representations for the functions $|\hat{\mu}_j(y)|$, $j = 1, 2$. Put $f(y) = -\ln|\hat{\mu}_1(y)|$, $g(y) = -\ln|\hat{\mu}_2(y)|$. We deduce from equation (7.20) that the functions $f(y)$ and $g(y)$ satisfy the equation

$$f(u+v) + g(u-v) = P(u) + P(v), \quad u, v \in Y, \tag{8.12}$$

where $P(y) = f(y) + g(y)$. Apply the finite-difference method for solving equation (8.12). Arguing as in the proof of Lemma 8.2 we get that the function $f(y)$ satisfies the equation

$$\Delta_{2v}\Delta_w^2 f(u) = 0, \quad v, w, u \in Y. \tag{8.13}$$

We note that by Theorem 1.9.5, $\overline{Y^{(2)}} = A(Y, X_{(2)})$. Then by Lemma 7.13 the restrictions of the characteristic functions $\hat{\mu}_j(y)$ to the subgroup $\overline{Y^{(2)}} = \mathbb{R} \times \mathbb{Z}^{(2)}$ are of the form

$$\mu_1(y) = (x_1, y)\exp\{-\varphi(y)\}, \quad \mu_2(y) = (x_2, y)\exp\{-\varphi(y)\}, \quad y \in \overline{Y^{(2)}},$$

where $x_j \in X$, and $\varphi(y)$ is a continuous nonnegative function on the subgroup $\overline{Y^{(2)}}$ satisfying equation 2.16 (ii). Taking into account Remark 5.12 we find

$$f(s,n) = g(s,n) = \sigma s^2 + 2\beta sn + \lambda n^2, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}^{(2)}. \tag{8.14}$$

Setting in equation (8.13) $w = v$, rewrite the obtained equation in the form

$$f(u+4v) - 2f(u+3v) + 2f(u+v) - f(u) = 0, \quad u, v \in Y.$$

Setting here $u = (s,n)$, $v = (0,1)$ we find

$$f(s,n+4) - 2f(s,n+3) + 2f(s,n+1) - f(s,n) = 0, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.15}$$

Put $f_s(n) = f(s,n)$, $s \in \mathbb{R}$, and rewrite equation (8.15) in the form

$$f_s(n+4) - 2f_s(n+3) + 2f_s(n+1) - f_s(n) = 0, \quad n \in \mathbb{Z}.$$

Solving this equation in the same way as equation (8.6) we obtain

$$f_s(n) = a_1(s) + b_1(s)n + c_1(s)n^2 + d_1(s)(-1)^n, \quad n \in \mathbb{Z},$$

where $a_1(s)$, $b_1(s)$, $c_1(s)$, $d_1(s)$ are some functions in s. Representation (8.3) for the function $f_s(n) = f(s,n)$ on the subgroup $\mathbb{R} \times \mathbb{Z}^{(2)}$ implies that $a_1(s) + d_1(s) = \sigma s^2$, $b_1(s) = 2\beta s$, $c_1(s) = \lambda$. Hence

$$f(s,n) = \sigma s^2 + 2\beta sn + \lambda n^2 - d_1(s)(1 - (-1)^n), \quad (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.16}$$

We find similarly

$$g(s,n) = \sigma s^2 + 2\beta sn + \lambda n^2 - d_2(s)(1 - (-1)^n), \quad (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.17}$$

Let us prove now that $d_1(s) = -d_2(s) = \text{const}$.

Substituting $u = (s,0)$, $v = (s',0)$ in equation 7.1 (i) we get

$$\hat{\mu}_1(s+s',0)\hat{\mu}_2(s-s',0) = \hat{\mu}_1(s,0)\hat{\mu}_1(s',0)\hat{\mu}_2(s,0)\hat{\mu}_2(-s',0), \quad s \in \mathbb{R}.$$

Taking into account (8.16) and (8.17) it follows from the Kac–Bernstein theorem that the solutions of this equation are of the form

$$\hat{\mu}_j(s,0) = \exp\{-\sigma s^2 + i t_j s\}, \quad s \in \mathbb{R},\ \sigma \ge 0,\ t_j \in \mathbb{R}. \tag{8.18}$$

Putting $u = (0,n)$, $v = (0,n')$ in equation 7.1 (i) we find that

$$\hat{\mu}_1(0,n+n')\hat{\mu}_2(0,n-n') = \hat{\mu}_1(0,n)\hat{\mu}_1(0,n')\hat{\mu}_2(0,n)\hat{\mu}_2(0,-n'), \quad n \in \mathbb{Z}. \tag{8.19}$$

In view of (8.16) and (8.17) it follows from Lemma 8.2 that all solutions of equation (8.19) are of the form

$$\hat{\mu}_1(0,n) = \exp\{-\lambda n^2 + i n\theta_1 + \kappa(1 - (-1)^n)\}, \quad n \in \mathbb{Z}, \tag{8.20}$$

$$\hat{\mu}_2(0,n) = \exp\{-\lambda n^2 + i n\theta_2 - \kappa(1 - (-1)^n)\}, \quad n \in \mathbb{Z}, \tag{8.21}$$

where $\lambda \ge 0$, $0 \le \theta_j < 2\pi$, $\kappa = d_1(0) = -d_2(0)$.

Substitute $u = (s,0)$, $v = (0,-n)$ in equation (7.20). Taking into account (8.18), (8.20), and (8.21) we obtain from the resulting equation

$$|\hat{\mu}_1(s,-n)||\hat{\mu}_2(s,n)| = \exp\{-2\sigma s^2 - 2\lambda n^2\}, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}.$$

Thus

$$f(s,-n) + g(s,n) = 2\sigma s^2 + 2\lambda n^2, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}.$$

Substituting here expressions (8.16) and (8.17) we find $(d_1(s)+d_2(s))(1-(-1)^n) = 0$. Thus

$$d_1(s) = -d_2(s), \quad s \in \mathbb{R}. \tag{8.22}$$

Put $u = (s_1,n_1)$, $v = (s_2,n_2)$ in (8.12) and substitute expressions (8.16) and (8.17) for the functions $f(s,n)$ and $g(s,n)$ into (8.12). Rearranging the obtained equality we get

$$\begin{aligned}&(d_1(s_1+s_2) + d_2(s_1-s_2))(1 - (-1)^{n_1+n_2}) \\ &\quad = (d_1(s_1) + d_2(s_1))(1 - (-1)^{n_1}) + (d_1(s_2) + d_2(-s_2))(1 - (-1)^{n_2}).\end{aligned}$$

Setting here $n_1 = 1, n_2 = 0$ and taking into account (8.22) we obtain $d_1(s_1 + s_2) = d_1(s_1 - s_2)$. Substituting here $s_1 = s_2 = s/2$ we find

$$d_1(s) = d_1(0) = \kappa. \tag{8.23}$$

Finally from (8.16), (8.17), (8.22), and (8.23) we obtain

$$f(s,n) = \sigma s^2 + 2\beta sn + \lambda n^2 - \kappa(1 - (-1)^n), \quad (s,n) \in \mathbb{R} \times \mathbb{Z},$$
$$g(s,n) = \sigma s^2 + 2\beta sn + \lambda n^2 + \kappa(1 - (-1)^n), \quad (s,n) \in \mathbb{R} \times \mathbb{Z}.$$

Hence

$$|\hat{\mu}_1(s,n)| = \exp\{-\sigma s^2 - 2\beta sn - \lambda n^2 + \kappa(1 - (-1)^n)\}, \quad (s,n) \in \mathbb{R} \times \mathbb{Z},$$
$$|\hat{\mu}_2(s,n)| = \exp\{-\sigma s^2 - 2\beta bsn - \lambda n^2 - \kappa(1 - (-1)^n)\}, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}.$$

Put

$$l_1(y) = \hat{\mu}_1(y)/|\hat{\mu}_1(y)|, \quad l_2(y) = \hat{\mu}_2(y)/|\hat{\mu}_2(y)|, \quad y \in Y,$$

and prove that the functions $l_j(y)$, $j = 1, 2$, are characters of the group Y. We note that

$$|l_j(y)| = 1, \quad l_j(-y) = \overline{l_j(y)}, \quad l_j(0) = 1, \quad y \in Y,\ j = 1, 2.$$

Since the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i), the functions $l_j(y)$ also satisfy equation 7.1 (i), i.e.,

$$l_1(u + v)l_2(u - v) = l_1(u)l_1(v)l_2(u)l_2(-v), \quad u, v \in Y. \tag{8.24}$$

Put $a_j(y) = a_j(s,n) = \exp\{it_j s + i\theta_j n\}$ and $\tilde{l}_j(y) = l_j(y)/a_j(y)$, $j = 1, 2$. Taking into account that the functions $a_j(y)$ satisfy equation (8.24), the functions $\tilde{l}_j(y)$ also satisfy equation (8.24), i.e.,

$$\tilde{l}_1(u + v)\tilde{l}_2(u - v) = \tilde{l}_1(u)\tilde{l}_1(v)\tilde{l}_2(u)\tilde{l}_2(-v), \quad u, v \in Y. \tag{8.25}$$

It follows from (8.18), (8.20), and (8.21) that

$$\tilde{l}_j(s, 0) = \tilde{l}_j(0, n) = 1, \quad j = 1, 2,\ (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.26}$$

Putting $u = (0, n)$, $v = (s, 0)$ and then $u = (s, 0)$, $v = (0, n)$ into equation (8.25) and taking into account (8.26) we find

$$\tilde{l}_1(s,n)\tilde{l}_2(-s,n) = 1, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}, \tag{8.27}$$
$$\tilde{l}_1(s,n)\tilde{l}_2(s,-n) = 1, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.28}$$

Multiplying (8.27) and (8.28) and using the equality $\tilde{l}_2(y)\tilde{l}_2(-y) = 1$ we arrive at

$$\tilde{l}_1^{\,2}(s,n) = 1, \quad (s,n) \in \mathbb{R} \times \mathbb{Z}. \tag{8.29}$$

Putting $u = v = (s/2, n)$ into equation (8.25) and considering (8.29) we get

$$\widetilde{l_1}(s, 2n) = 1, \quad (s, n) \in \mathbb{R}\times\mathbb{Z}. \tag{8.30}$$

Put $u = (s, n)$, $v = (0, n)$ in equation (8.25) and make use of (8.26) and (8.30). We find

$$\widetilde{l_1}(s, n)\widetilde{l_2}(s, n) = 1, \quad (s, n) \in \mathbb{R}\times\mathbb{Z}. \tag{8.31}$$

Substituting $u = (s/2, n)$, $v = (s/2, 0)$ into equation (8.25) and taking into account (8.26) and (8.31) we arrive at

$$\widetilde{l_1}(s, n) = 1, \quad (s, n) \in \mathbb{R}\times\mathbb{Z}. \tag{8.32}$$

We conclude from (8.31) and (8.32) that

$$\widetilde{l_2}(s, n) = 1, \quad (s, n) \in \mathbb{R}\times\mathbb{Z}.$$

Returning to the original characteristic functions $\hat\mu_j(s, n)$, we obtain the representations

$$\hat\mu_1(s, n) = \exp\{-\sigma s^2 - 2\beta sn - \lambda n^2 + \kappa(1 - (-1)^n)\} + it_1 s + in\theta_1\}, \tag{8.33}$$

$$\hat\mu_2(s, n) = \exp\{-\sigma s^2 - 2\beta bsn - \lambda n^2 - \kappa(1 - (-1)^n) + it_2 s + in\theta_2\}, \tag{8.34}$$

$(s, n) \in \mathbb{R}\times\mathbb{Z}$. By Lemma 8.3, $a_1(s, n) = \exp\{\kappa(1 - (-1)^n)\}$, $(s, n) \in \mathbb{R}\times\mathbb{Z}$, is the characteristic function of the signed measure π_1 which is defined by 8.3 (i). Obviously, the function $b(s, n) = \exp\{-\sigma s^2 - 2\beta sn - \lambda n^2\}$, $(s, n) \in \mathbb{R}\times\mathbb{Z}$, is the characteristic function of a Gaussian distribution $\gamma \in \Gamma(X)$. It follows from (8.33) that

$$\hat\mu_1(s, n) = \hat\gamma(s, n)\pi_1(s, n)\exp\{it_1 s + in\theta_1\}, \quad (s, n) \in \mathbb{R}\times\mathbb{Z}. \tag{8.35}$$

Taking into account 2.7 (b) and (c) we find from (8.35) $\mu_1 = \gamma * \pi_1 * E_{x_1}$, where $x_1 = (t_1, e^{i\theta_1}) \in X$. Arguing as above we obtain the representation for the distribution μ_2. So, we have proved that in the case when $N = \mathbb{R}\times\mathbb{Z}$, assertion (i) of the theorem holds.

2. $N = \mathbb{R}\times\mathbb{Z}^{(p)}$, where $p \neq 1$. We deduce from (7.4) that the subgroup N has the property: if $2y \in N$, then $y \in N$. This implies that p is an odd number.

Let us verify that $N_1 = N_2 = N$. Note that in view of 2.7 (d) we have $|\hat\mu_1(-v)| = |\hat\mu_1(v)|$. Replacing v by $-v$ in (7.20) we find

$$|\hat\mu_1(u - v)||\hat\mu_2(u + v)| = |\hat\mu_1(u)||\hat\mu_1(v)|\hat\mu_2(u)|\hat\mu_2(v)|, \quad u, v \in Y. \tag{8.36}$$

Observe that the equality

$$|\hat\mu_1(t + t', m + m')||\hat\mu_2(t - t', m - m')| \\ = |\hat\mu_1(t - t', m - m')||\hat\mu_2(t + t', m + m')|, \quad (t, m), (t', m') \in \mathbb{R}\times\mathbb{Z},$$

follows from (7.20) and (8.36). This implies

$$|\hat\mu_1(s, n)||\hat\mu_2(s', n')| = |\hat\mu_1(s', n')||\hat\mu_2(s, n)|, \tag{8.37}$$

where s, s' are arbitrary real numbers, and n, n' are integers with the same parity. Assume that there is an element $(s_0, n_0) \in N_1$ such that $(s_0, n_0) \notin N_2$. Let $(s', n') \in N$, where n' and n_0 have the same parity. Then (8.37) implies that $\hat{\mu}_2(s', n') = 0$, contrary to the assumption. Hence $N_1 \subset N_2$. The same reasoning shows that $N_2 \subset N_1$. So, $N_1 = N_2 = N$.

Consider the restriction of equation 7.1 (i) to the subgroup N. In view of $N = \mathbb{R} \times \mathbb{Z}^{(p)} \cong \mathbb{R} \times \mathbb{Z}$ we obtain from (8.33) and (8.34) the following representation of solutions of equation 7.1 (i):

$$\hat{\mu}_1(s,n) = \begin{cases} \exp\{-\sigma s^2 - 2\beta sn - \lambda n^2 + \kappa(1-(-1)^n) + it_1 s + in\theta_1\} & \text{if } n \in \mathbb{Z}^{(p)}, \\ 0 & \text{if } n \notin \mathbb{Z}^{(p)}, \end{cases}$$

$$\hat{\mu}_2(s,n) = \begin{cases} \exp\{-\sigma s^2 - 2\beta sn - \lambda n^2 - \kappa(1-(-1)^n) + it_1 s + in\theta_1\} & \text{if } n \in \mathbb{Z}^{(p)}, \\ 0 & \text{if } n \notin \mathbb{Z}^{(p)}. \end{cases}$$

Note that $A(\mathbb{R} \times \mathbb{Z}, \mathbb{Z}(p)) = \mathbb{R} \times \mathbb{Z}^{(p)}$. Then by 2.14 (i) the characteristic function of the Haar distribution $m_{\mathbb{Z}(p)}$ is of the form

$$\widehat{m}_{\mathbb{Z}(p)}(s,n) = \begin{cases} 1 & \text{if } n \in \mathbb{Z}^{(p)}, \\ 0 & \text{if } n \notin \mathbb{Z}^{(p)}. \end{cases}$$

Reasoning as in the final part of the proof of the theorem in case 1, we get that in the case when $N = \mathbb{R} \times \mathbb{Z}^{(p)}$, where $p \neq 1$, assertion (ii) of the theorem holds.

3. $N = \mathbb{R}$. It follows from equation (7.6) that $|\hat{\mu}_1(s, 2n)| = |\hat{\mu}_2(s, 2n)|$ for all $s \in \mathbb{R}$, $n \in \mathbb{Z}$. Since $N = \mathbb{R}$, we have $|\hat{\mu}_1(s, 2n)| = |\hat{\mu}_2(s, 2n)| = 0$ for all $s \in \mathbb{R}$, $n \in \mathbb{Z}$, $n \neq 0$. Assume that there exist $s_0 \in \mathbb{R}$ and an odd integer $n_0 \in \mathbb{Z}$ such that $\hat{\mu}_1(s_0, n_0) \neq 0$. Then $\hat{\mu}_2(s_0, n_0) = 0$ and (8.37) implies that $\hat{\mu}_2(s', n') = 0$ for all $s' \in \mathbb{R}$ and any odd integer n'. In view of (8.18) it follows from this that

$$\hat{\mu}_1(s, 2n) = \begin{cases} \exp\{-\sigma s^2 + it_1 s\} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$

$$\hat{\mu}_2(s, n) = \begin{cases} \exp\{-\sigma s^2 + it_2 s\} & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Observe that $A(\mathbb{R} \times \mathbb{Z}, \mathbb{T}) = \mathbb{R}$. Then by 2.14 (i) the characteristic function of the Haar distribution $m_{\mathbb{T}} \in \mathrm{M}^1(X)$ is of the form

$$\widehat{m}_{\mathbb{T}}(s,n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The function $\lambda(s,n) = \exp\{-\sigma s^2\}$, $(s,n) \in \mathbb{R} \times \mathbb{Z}$, is the characteristic function of a Gaussian distribution γ supported on $\mathbb{R}$. Hence

$$\begin{aligned} \hat{\mu}_1(s, 2n) &= \hat{\gamma}(s,n)\widehat{m}_{\mathbb{T}}(s,n)\exp\{it_1 s\}, \\ \hat{\mu}_2(s, n) &= \hat{\gamma}(s,n)\widehat{m}_{\mathbb{T}}(s,n)\exp\{it_2 s\}. \end{aligned} \tag{8.38}$$

Taking into account 2.7 (b), 2.7 (c), and 8.4 we find from (8.38) that $\tau(\mu_1) = \gamma * m_{\mathbb{T}} * E_{(t_1,1)}$, and $\mu_2 = \gamma * m_{\mathbb{T}} * E_{(t_2,1)}$, where τ is defined by (8.11).

Reasoning similarly in the case when there exists an odd integer $n_0 \in \mathbb{Z}$ such that $\hat{\mu}_2(s, n_0) \neq 0$ we find that $\mu_1 = \gamma * m_{\mathbb{T}} * E_{(t_1,1)}$ and $\tau(\mu_2) = \gamma * m_{\mathbb{T}} * E_{(t_2,1)}$. Thus we have proved that in the case when $N = \mathbb{R}$, assertion (iii) of the theorem holds. □

Corollary 8.6. *Let ξ_1 and ξ_2 be independent random variables with values in the circle group $\mathbb{T}$ and distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then one of the following statements is true:*

- (i) *$\mu_j = \gamma * \pi_j * E_{x_j}$, where $\gamma \in \Gamma(\mathbb{T})$, $x_j \in \mathbb{T}$, and π_j are signed measures concentrated on the subgroup $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_1$;*
- (ii) *$\mu_j = \gamma * m_{\mathbb{Z}(n)} * \pi_j * E_{x_j}$, where $\gamma \in \Gamma(\mathbb{T})$, n is an odd number, $x_j \in \mathbb{T}$, and π_j are signed measures concentrated on the subgroup $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_1$;*
- (iii) *either the random variables $2\xi_1$ and ξ_2 are identically distributed with distribution $m_{\mathbb{T}}$ or the random variables ξ_1 and $2\xi_2$ are identically distributed with distribution $m_{\mathbb{T}}$.*

If we assume additionally that the characteristic functions of the distributions μ_j do not vanish, then only case (i) *is possible.*

8.7. Let $\boldsymbol{a} = (a_0, a_1, \dots, a_n, \dots)$ be a fixed but arbitrary infinite sequence of integers, where each of a_n is greater than 1, let $\Sigma_{\boldsymbol{a}}$ be the $\boldsymbol{a}$-adic solenoid (see 1.2 (g)). Consider now the case when independent random variables ξ_1 and ξ_2 take values in the group $X = \Sigma_{\boldsymbol{a}}$ and have distributions μ_1 and μ_2. If $X_{(2)} = \{0\}$, then by Theorem 7.10 the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that $\mu_j = \gamma * m_K * E_{x_j}$, where $\gamma \in \Gamma(X)$, $x_j \in X$, and K is a compact Corwin subgroup of the group X. We will study the case when $X_{(2)} \neq \{0\}$. The following theorem holds.

Theorem 8.8. *Let $X = \Sigma_{\boldsymbol{a}}$ be an $\boldsymbol{a}$-adic solenoid such that $X_{(2)} \neq \{0\}$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then one of the following statements is true:*

- (i) *$\mu_j = \gamma * m_K * E_{x_j}$, where $\gamma \in \Gamma(X)$, $x_j \in X$, and K is a compact Corwin subgroup of X;*
- (ii) *$\mu_j = \gamma * m_K * \pi_j * E_{x_j}$, where $\gamma \in \Gamma(X)$, $x_j \in X$, K is a compact Corwin subgroup of X, and π_j are signed measures concentrated on the subgroup $X_{(2)}$ such that $\pi_1 * \pi_2 = E_0$;*
- (iii) *either the random variables $2\xi_1$ and ξ_2 are identically distributed with distribution m_X or the random variables ξ_1 and $2\xi_2$ are identically distributed with distribution m_X.*

Proof. We note that the character group $Y = \Sigma_{\boldsymbol{a}}^*$ is topologically isomorphic to a subgroup of $\mathbb{Q}$ of the form $H_{\boldsymbol{a}}$, where $H_{\boldsymbol{a}} = \{\frac{m}{a_0 a_1 \dots a_n} : n = 0, 1, \dots; m \in \mathbb{Z}\}$ (see 1.10 (e)). To avoid introducing new notation we will assume that $Y = H_{\boldsymbol{a}}$. By

Theorem 1.9.5, $X_{(2)} = A(X, Y^{(2)})$. It follows from this that the condition $X_{(2)} \neq \{0\}$ is fulfilled if and only if $Y \neq Y^{(2)}$. It is easy to see that in this case the group Y contains only elements of the form $p/2^k q$, where $p \in \mathbb{Z}$, q is an odd number, and k does not exceed a positive integer. Replacing the group Y by its isomorphic group we can assume without loss of generality that every element of the group Y is of the form p/q, where $p \in \mathbb{Z}$, and q is an odd number. Obviously, we can also assume that the greatest common factor of all numerators p of the elements p/q of the group Y is equal to 1. We conclude from this that the group Y has the property: if $p/q \in Y$, then $1/q \in Y$.

By Lemma 7.1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i). Hence they also satisfy equations (7.4), (7.6), and (7.20). As in the proof of Theorem 8.5, consider the sets

$$N_1 = \{y \in Y : \hat{\mu}_1(y) \neq 0\}, \quad N_2 = \{y \in Y : \hat{\mu}_2(y) \neq 0\},$$

and the subgroup $N = N_1 \cap N_2$. There are two possibilities for N: $N \neq \{0\}$ and $N = \{0\}$.

1. $N \neq \{0\}$. Note that it follows from (7.4) that the subgroup N has the property: if $2y \in N$, then $y \in N$. This implies that the subgroup N contains elements of the form p/q, where p is an odd integer. Note also that equation (8.36) follows from equation (7.20). Let us check that $N_1 = N_2 = N$. For this purpose observe that it follows from equations (7.20) and (8.36) that

$$|\hat{\mu}_1(u+v)||\hat{\mu}_2(u-v)| = |\hat{\mu}_1(u-v)||\hat{\mu}_2(u+v)|, \quad u, v \in Y.$$

Hence for any elements $p/q, p'/q' \in Y$ such that the integers p and p' have the same parity the equality

$$|\hat{\mu}_1(p/q)||\hat{\mu}_2(p'/q')| = |\hat{\mu}_1(p'/q')||\hat{\mu}_2(p/q)| \tag{8.39}$$

is true. Assume that there exists an element $p_0/q_0 \in N_1$ such that $p_0/q_0 \notin N_2$. Let $p'/q' \in N$, where p' has the same parity as p_0. Then it follows from (8.39) that $\hat{\mu}_2(p'/q') = 0$, contrary to the assumption. Hence $N_1 \subset N_2$. The same reasoning shows that $N_2 \subset N_1$, so $N_1 = N_2 = N$. Since N is a subgroup of Y, and Y is a subgroup of $\mathbb{Q}$, N is also a subgroup of $\mathbb{Q}$. Two cases are possible: $N \ncong \mathbb{Z}$ and $N \cong \mathbb{Z}$.

A. $N \ncong \mathbb{Z}$. It follows from the Pontryagin duality theorem and 1.10 (e) that the character group N^* is topologically isomorphic to a group $\Sigma_{\boldsymbol{b}}$. Denote by $g_j(y)$ the restriction of the characteristic function $\hat{\mu}_j(y)$ to the subgroup N. By the Bochner theorem, $g_j(y)$ are the characteristic functions of some distributions $\lambda_j \in \mathrm{M}^1(\Sigma_{\boldsymbol{b}})$. The characteristic functions $g_j(y)$ also satisfy equation 7.1 (i) and $g_j(y) \neq 0$ for all $y \in N$. Since the group $\Sigma_{\boldsymbol{b}}$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, it follows from Lemmas 7.1 and 7.9 that $\lambda_j \in \Gamma(\Sigma_{\boldsymbol{b}})$, and the characteristic functions $g_j(y)$ are of the form

$$g_1(y) = l_1(y)\exp\{-\varphi(y)\}, \quad g_2(y) = l_2(y)\exp\{-\varphi(y)\}, \quad y \in N,$$

where $l_j(y)$ are characters of the group N, and $\varphi(y)$ is a nonnegative function on N satisfying equation 2.16 (ii).

It is easy to see that the function $\varphi(y)$ is of the form $\varphi(y) = \sigma y^2$, $y \in N$. By Theorem 1.9.2 the characters $l_j(y)$ are represented in the form $l_j(y) = (x_j, y)$, $x_j \in \Sigma_{\boldsymbol{a}}$, $y \in N$. Hence we have

$$g_1(y) = (x_1, y)\exp\{-\sigma y^2\}, \quad g_2(y) = (x_2, y)\exp\{-\sigma y^2\}, \quad y \in N.$$

We deduce from this that the characteristic functions $\hat{\mu}_j(y)$ are represented in the form

$$\hat{\mu}_1(y) = \begin{cases} (x_1, y)\exp\{-\sigma y^2\} & \text{if } y \in N, \\ 0 & \text{if } y \notin N, \end{cases}$$
$$\hat{\mu}_2(y) = \begin{cases} (x_2, y)\exp\{-\sigma y^2\} & \text{if } y \in N, \\ 0 & \text{if } y \notin N. \end{cases} \tag{8.40}$$

Put $K = A(X, N)$ and observe that the subgroup N has the property: if $2y \in N$, then $y \in N$. By Proposition 7.4 this yields that K is a Corwin group. By Theorem 1.9.1, $N = A(Y, K)$ and in view of 2.14 (i) the characteristic function $\widehat{m}_K(y)$ is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in N, \\ 0 & \text{if } y \notin N. \end{cases} \tag{8.41}$$

The function $\exp\{-\sigma y^2\}$, $y \in Y$, is the characteristic function of a Gaussian distribution $\gamma \in \Gamma(X)$. It follows from (8.40) and (8.41) that $\hat{\mu}_j(y) = \hat{\gamma}(y)\widehat{m}_K(y)(x_j, y)$. By 2.7 (b) and 2.7 (c) this yields that $\mu_j = \gamma * m_K * E_{x_j}$, $j = 1, 2$.

So, we have proved statement (i) of the theorem in the case when N is not topologically isomorphic to the group $\mathbb{Z}$.

B. Let $N \cong \mathbb{Z}$ and let r_0 be a generator of the group N, i.e., $N = \{nr_0 : n \in \mathbb{Z}\}$, where $r_0 = p_0/q_0 \in Y$. The subgroup N has the property: if $2y \in N$, then $y \in N$. Hence p_0 is an odd integer. Moreover, q_0 is also an odd integer. We conclude from Lemma 8.2 that the restrictions of the characteristic functions $\hat{\mu}_j(y)$ to the subgroup N are of the form

$$\hat{\mu}_1(y) = \exp\{-\lambda n^2 + in\theta_1 + \kappa(1 - (-1)^n)\},$$
$$\hat{\mu}_2(y) = \exp\{-\lambda n^2 + in\theta_2 - \kappa(1 - (-1)^n)\}, \tag{8.42}$$

where $y = nr_0, n \in \mathbb{Z}, \lambda \ge 0, 0 \le \theta_j < 2\pi, \kappa \in \mathbb{R}$. Since $r_0 = p_0/q_0$, where p_0, q_0 are odd integers, we have $(-1)^n = (-1)^{nr_0} = (-1)^y$ and we can rewrite (8.42) in the form

$$\hat{\mu}_1(y) = \exp\{-\lambda' y^2 + iy\theta_1' + \kappa(1 - (-1)^y)\},$$
$$\hat{\mu}_2(y) = \exp\{-\lambda' y^2 + iy\theta_2' - \kappa(1 - (-1)^y)\},$$

where $\lambda' = \lambda/r_0^2$, $\theta_j' = \theta_j/r_0$. It follows from this the representations

$$\hat{\mu}_1(y) = \begin{cases} \exp\{-\lambda' y^2 + iy\theta_1' + \kappa(1-(-1)^y)\} & \text{if } y \in N, \\ 0 & \text{if } y \notin N, \end{cases} \tag{8.43}$$

$$\hat{\mu}_2(y) = \begin{cases} \exp\{-\lambda' y^2 + iy\theta_2' - \kappa(1-(-1)^y)\} & \text{if } y \in N, \\ 0 & \text{if } y \notin N, \end{cases} \tag{8.44}$$

where $\lambda' \geq 0$, $\kappa \in \mathbb{R}$, $0 \leq \theta_j' < 2\pi/r_0$. By Theorem 1.9.5, $A(Y, X_{(2)}) = Y^{(2)}$. Then by Theorem 1.9.2, $(X_{(2)})^* \cong Y/A(Y, X_{(2)}) = Y/Y^{(2)} \cong \mathbb{Z}(2)$. Hence $X_{(2)} = \{0, x_0\}$, where x_0 is an element of order 2 in the group X. We recall that any element of the group Y is of the form $y = p/q$, where $p \in \mathbb{Z}$ and q is an odd integer. Therefore $a_1(y) = \exp\{\kappa(1-(-1)^y\}$, $y \in Y$, is the characteristic function of the signed measure

$$\pi_1 = \frac{1}{2}(1+e^{2\kappa})E_0 + \frac{1}{2}(1-e^{2\kappa})E_{x_0};$$

moreover $b(y) = \exp\{-\lambda y^2\}$, $y \in Y$, is the characteristic function of a Gaussian distribution $\gamma \in \Gamma(X)$. The function $l_1(y) = \exp\{iy\theta_1'\}$ is a character of the subgroup N. By Theorem 1.9.2 the function $l_1(y)$ is represented in the form $l_1(y) = (x_1, y)$, $x_1 \in X$. We deduce from (8.43) that $\hat{\mu}_1(y) = \hat{\gamma}(y)\widehat{m}_K(y)\hat{\pi}_1(y)(x_1, y)$. By 2.7 (b) and 2.7 (c) this yields that $\mu_1 = \gamma * m_K * \pi_1 * E_{x_1}$. Arguing as above we obtain from (8.44) the representation for the distribution μ_2. In this case

$$\pi_2 = \frac{1}{2}(1+e^{-2\kappa})E_0 + \frac{1}{2}(1-e^{-2\kappa})E_{x_0}.$$

It is easily seen that $\pi_1 * \pi_2 = E_0$. So, we have proved statement (ii) of the theorem in the case when N is a subgroup of $\mathbb{Q}$ topologically isomorphic to the group $\mathbb{Z}$.

2. $N = \{0\}$. Since $N = \{0\}$, it follows from (7.6) that $\hat{\mu}_1(2p/q) = \hat{\mu}_2(2p/q) = 0$ for all $p/q \in Y$, $p \neq 0$. Assume that there is an element p_0/q_0, where p_0 is an odd integer such that $\hat{\mu}_1(p_0/q_0) \neq 0$. Then $\hat{\mu}_2(p_0/q_0) = 0$ and it follows from (8.39) that $\hat{\mu}_2(p'/q') = 0$ for all $p'/q' \in Y$, where p' is an odd integer. Thus we get that

$$\hat{\mu}_1(2p/q) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases} \quad \hat{\mu}_2(p/q) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0 \end{cases}$$

for all $p/q \in Y$. In view of (2.2) and 2.7 (b) the obtained representations of the characteristic functions $\hat{\mu}_j(y)$ show that the random variables $2\xi_1$ and ξ_2 are identically distributed with distribution m_X. In the case when there is an element p_0/q_0, where p_0 is an odd integer such that $\hat{\mu}_2(p_0/q_0) \neq 0$, we reason similarly. Then we conclude that the random variables ξ_1 and $2\xi_2$ are identically distributed with distribution m_X. So, we have proved statement (iii) of the theorem in the case when $N = \{0\}$. The theorem is completely proved. □

Remark 8.9. We note that if $\kappa \neq 0$, then one of the signed measures π_j appearing in Theorems 8.5 and 8.8 in fact is a distribution.

Remark 8.10. It should be noted that if ξ_1 and ξ_2 are independent random variables with values in the group $X = \mathbb{R} \times \mathbb{T}$ and distributions μ_j such as in statements (i)–(iii) of Theorem 8.5, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 7.1 (i), and hence by Lemma 7.1, $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent.

The analogous assertion is also true for the random variables ξ_1 and ξ_2 with values in the group $X = \Sigma_{\boldsymbol{a}}$ and distributions μ_j such as in statements (i)–(iii) of Theorem 8.8.

9 Gaussian distributions in the sense of Bernstein

Let X be a second countable locally compact Abelian group, Y be its character group. In this section we study distributions of independent identically distributed random variables taking values in X and having independent the sum and the difference. Such distributions are called Gaussian distributions in the sense of Bernstein. We describe groups X with the property: any Gaussian distribution in the sense of Bernstein on a group X is invariant with respect to some compact subgroup K of X and under the natural homomorphism $X \mapsto X/K$ induces on the factor group X/K a Gaussian distribution.

Definition 9.1. A distribution μ on a group X is called a *Gaussian distribution in the sense of Bernstein* if μ has the following property: if ξ_1 and ξ_2 are independent identically distributed random variables with values in X and distribution μ, then their sum and difference are independent.

We denote by $\Gamma_B(X)$ the set of Gaussian distributions in the sense of Bernstein on the group X.

Lemma 9.2. *A distribution $\mu \in \mathrm{M}^1(X)$ belongs to the class $\Gamma_B(X)$ if and only if the characteristic function $\hat{\mu}(y)$ satisfies the equation*

$$\text{(i)} \quad \hat{\mu}(u+v)\hat{\mu}(u-v) = \hat{\mu}^2(u)|\hat{\mu}(v)|^2, \quad u, v \in Y.$$

Proof. The assertion of the lemma follows from 2.7 (d) and Lemma 7.1, if we suppose in Lemma 7.1 that $\mu_1 = \mu_2 = \mu$. □

9.3. Equation 9.2 (i) yields the inclusion $\Gamma(X) \subset \Gamma_B(X)$. Obviously, the class $I_B(X)$ defined in 7.1 consists of idempotent distributions which belong to the class $\Gamma_B(X)$, i.e., $I_B(X) = \Gamma_B(X) \cap I(X)$. As has been proved in Proposition 7.4, $m_K \in I_B(X)$ if and only if K is a compact Corwin group. It follows from 2.7 (c) and equation 9.2 (i) that the set $\Gamma_B(X)$ is a subsemigroup of $\mathrm{M}^1(X)$. This implies the inclusion

$$\Gamma(X) * I_B(X) \subset \Gamma_B(X). \tag{9.1}$$

The main problem solved in this section is the following: to describe all groups X for which

$$\Gamma(X) * I_B(X) = \Gamma_B(X). \tag{9.2}$$

It is obvious that if $\mu \in \Gamma(X) * I_B(X)$, then μ is invariant with respect to some compact Corwin subgroup K of X and under the natural homomorphism $X \mapsto X/K$ induces on the factor group X/K a Gaussian distribution.

To solve this problem we will need some lemmas.

Lemma 9.4. *Let $\mu \in \Gamma_B(X)$ and assume that the characteristic function $\hat{\mu}(y)$ does not vanish. Then $\hat{\mu}(y)$ can be represented in the form*

$$\text{(i)} \quad \hat{\mu}(y) = l(y)\exp\{-\varphi(y)\}, \quad y \in Y,$$

where $l(y)$ is a continuous function satisfying the equation

$$\text{(ii)} \quad l(u+v)l(u-v) = l^2(u), \quad u, v \in Y,$$

and the conditions

$$\text{(iii)} \quad l(-y) = \overline{l(y)}, \quad |l(y)| = 1, \quad l(0) = 1, \quad y \in Y,$$

and $\varphi(y)$ is a continuous nonnegative function satisfying equation 2.16 (ii).

Proof. By Lemma 9.2 the characteristic function $\hat{\mu}(y)$ satisfies equation 9.2 (i). We conclude from equation 9.2 (i) that

$$|\hat{\mu}(u+v)||\hat{\mu}(u-v)| = |\hat{\mu}(u)|^2|\hat{\mu}(v)|^2, \quad u, v \in Y.$$

Take the logarithm of both sides of this equality and put $\varphi(y) = -\ln|\hat{\mu}(y)|$. We get that $\varphi(y)$ is a continuous nonnegative function satisfying equation 2.16 (ii), and $|\hat{\mu}(y)| = \exp\{-\varphi(y)\}$. Set $l(y) = \hat{\mu}(y)/|\hat{\mu}(y)|$. It is obvious that the function $l(y)$ is continuous and satisfies conditions (iii). The function $l(y)$ also satisfies equation 9.2 (i). In view of (iii) this equation is transformed into equation (ii). □

Lemma 9.5. *The following conditions are equivalent for a group X:*

(i) *for any compact Corwin subgroup K of a group X the factor group X/K contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$;*

(ii) *the connected component of zero c_X of a group X contains no more than one element of order two, i.e., $|(c_X)_{(2)}| \le 2$. To put it in another way, either $(c_X)_{(2)} = \{0\}$ or $(c_X)_{(2)} \cong \mathbb{Z}(2)$.*

Proof. (i) $\Longrightarrow$ (ii). We will verify that if K is a compact Corwin subgroup of X, then

$$|K_{(2)}| \le 2. \tag{9.3}$$

Assertion (ii) follows directly from (9.3). Indeed, by Theorem 1.11.2, $c_X \cong \mathbb{R}^m \times B$, where B is a compact connected group. By Theorem 1.9.6, B is a Corwin group. We deduce from (9.3) that $|B_{(2)}| \le 2$. Hence $|(c_X)_{(2)}| \le 2$, i.e., assertion (ii) is true.

Put $L = K^*$. By Theorem 1.6.1, L is a discrete group. It follows from Theorem 1.9.5 that $A(L, K_{(2)}) = L^{(2)}$, hence Theorem 1.9.5 yields that

$$(K_{(2)})^* \cong L/A(L, K_{(2)}) = L/L^{(2)}. \tag{9.4}$$

If we prove that
$$|L/L^{(2)}| \le 2, \tag{9.5}$$
then it will follow from this that $(K_{(2)})^*$ is a finite group. Hence the group $K_{(2)}$ is also finite and the equality $|K_{(2)}| = |(K_{(2)})^*|$ holds. Thus the desired assertion will be proved.

Inequality (9.5) will be proved if we check that for any $y_1, y_2 \in L$ such that $y_1, y_2 \notin L^{(2)}$, the inclusion $y_1 - y_2 \in L^{(2)}$ is fulfilled. Assume the contrary. Then there exist elements $y_1, y_2 \in L$ such that $y_1, y_2, y_1 - y_2 \notin L^{(2)}$. Denote by M the subgroup of L generated by the elements y_1 and y_2. Since $K^{(2)} = K$, by Theorem 1.9.5, $L_{(2)} = 0$. Hence all nonzero elements of finite order in the group L have odd order. This implies that if $y \in L$, $y \notin L^{(2)}$, then y is an element of infinite order. Since M is a finite generated group, by Theorem 1.19.4, $M = M_1 \times M_2$, where M_j is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}(n_j)$. Therefore “a priori” there are the following possibilities: $M \cong \mathbb{Z}$, $M \cong \mathbb{Z} \times \mathbb{Z}(n)$, $M \cong \mathbb{Z}^2$. Consider each of these cases.

1. $M \cong \mathbb{Z}$. We have $y_1 = n_1 e$, $y_2 = n_2 e$, where $e \in L$, e is an element of infinite order, and n_1, n_2 are integers. Since $y_1, y_2 \notin L^{(2)}$, the integers n_1 and n_2 are odd. Therefore $y_1 - y_2 = (n_1 - n_2)e \in L^{(2)}$, contrary to the assumption. Thus case 1 is impossible.

2. $M \cong \mathbb{Z} \times \mathbb{Z}(n)$. We have $y_1 = n_1 e + l_1 a$, $y_2 = n_2 e + l_2 a$, where $e, a \in L$, e is an element of infinite order, a is an element of finite order n, and n_j, l_j are integers. As has been noted above, all nonzero elements of finite order in the group L have odd order. Hence $a = 2b$, $b \in L$. In view of $y_1, y_2 \notin L^{(2)}$ this implies that the integers n_1 and n_2 are odd. Hence $y_1 - y_2 = (n_1 - n_2)e + 2(l_1 - l_2)b \in L^{(2)}$, contrary to the assumption. Thus case 2 is also impossible.

3. $M \cong \mathbb{Z}^2$. Let $y \in L$ be such an element that $2y \in M$. Then
$$2y = k_1 y_1 + k_2 y_2, \tag{9.6}$$
where k_1, k_2 are integers. Assume that for any $y \in L$ such that $2y \in M$ the numbers k_1, k_2 in (9.6) are even. In view of $L_{(2)} = \{0\}$ it follows from (9.6) that $y \in M$. Then by Theorem 1.9.1 and Lemma 7.2 the annihilator $A(K, M)$ is a compact Corwin subgroup. By Theorems 1.9.1 and 1.9.2, $K/A(K, M) \cong M^*$. Since $M^* \cong T^2$ and $K/A(K, M)$ is a subgroup of $X/A(K, M)$, we have obtained a contradiction. Hence there is an element $y \in L$ such that $2y \in M$ and at least one of the numbers k_j in (9.6) is odd.

Assume that $k_1 = 2m_1 + 1$, $k_2 = 2m_2$, where m_j are integers. Then we have $y_1 = 2y - 2m_1 y_1 - 2m_2 y_2 \in L^{(2)}$ contrary to the assumption. Arguing as above we obtain that the case $k_1 = 2m_1$, $k_2 = 2m_2 + 1$ is also impossible. Assume that $k_1 = 2m_1 + 1, k_2 = 2m_2 - 1$. Then $y_1 - y_2 = 2y - 2m_1 y_1 - 2m_2 y_2 \in L^{(2)}$, contrary to the assumption. Thus case 3 is also impossible. We have proved that inequality (9.5) is fulfilled. Hence (ii) also holds.

(ii) $\Longrightarrow$ (i). First we will prove that if K is a compact Corwin subgroup of X and $L = K^*$, then (9.5) is true. By Theorems 1.9.2 and 1.9.3,
$$(c_K)^* \cong L/b_L. \tag{9.7}$$

It follows from (ii) that $|(c_K)_{(2)}| \leq 2$. Therefore

$$|((c_K)_{(2)})^*| \leq 2. \tag{9.8}$$

By Theorem 1.9.5, $A((c_K)^*, (c_K)_{(2)}) = (c_K)^{*(2)}$. In view of (9.7), Theorem 1.9.2 yields

$$((c_K)_{(2)})^* \cong (c_K)^*/A((c_K)^*, (c_K)_{(2)}) \cong (L/b_L)/(L/b_L)^{(2)}. \tag{9.9}$$

Consider the natural homomorphism $L \mapsto L/b_L$. We conclude from (9.8) and (9.9) that if $[y_1], [y_2] \in L/b_L$, $[y_1], [y_2] \notin (L/b_L)^{(2)}$, then

$$[y_1] - [y_2] \in (L/b_L)^{(2)}. \tag{9.10}$$

Observe that if $y \in L$ and $[y] \in (L/b_L)^{(2)}$, then $y \in L^{(2)}$. Indeed, let $[y] = 2[y']$, $[y'] \in L/b_L$. It follows from this that $y - 2y' = h$, $h \in b_L$. Since $K^{(2)} = K$, Theorem 1.9.5 implies that $L_{(2)} = \{0\}$. Hence all nonzero elements of subgroup b_L have odd order. Therefore $h = 2h'$, $h' \in b_L$ and $y = 2y' + 2h' \in L^{(2)}$. Assume that $y_1, y_2 \in L$, $y_1, y_2 \notin L^{(2)}$. Then $[y_1], [y_2] \in L/b_L$ and $[y_1], [y_2] \notin (L/b_L)^{(2)}$. Hence (9.10) is true, so that $y_1 - y_2 \in L^{(2)}$. Thus we have proved (9.5).

Let G be a compact Corwin subgroup of X and F be a compact Corwin subgroup of the factor group X/G. We will verify that $|F_{(2)}| \leq 2$. Then (i) will be proved because the two-dimensional torus $\mathbb{T}^2$ is a compact Corwin group and $|(\mathbb{T}^2)_{(2)}| = 4$. Let $p\colon X \mapsto X/G$ be the natural homomorphism. Put $K = p^{-1}(F)$. It is obvious that $F \cong K/G$. Since the groups G and F are compact, the group K is also compact. It is clear that K is a Corwin group. As has been shown above, (9.5) holds true. Since G is a Corwin group, it is easy to see that $A(L, G) \cap L^{(2)} \subset (A(L, G))^{(2)}$. Let $y_1, y_2 \in A(L, G)$, $y_1, y_2 \notin (A(L, G))^{(2)}$. Then $y_1, y_2 \notin L^{(2)}$ and it follows from (9.5) that $y_1 - y_2 \in L^{(2)}$. Hence $y_1 - y_2 \in (A(L, G))^{(2)}$. Taking into account that, by Theorem 1.9.2, $F^* \cong A(L, G)$, we have proved that $|F^*/(F^*)^{(2)}| \leq 2$. By Theorems 1.9.2 and 1.9.5, $(F_{(2)})^* \cong F^*/(F^*)^{(2)}$. So we obtain that $|(F_{(2)})^*| \leq 2$. Hence $(F_{(2)})^*$ is a finite group. Thus the group $F_{(2)}$ is also finite and $|F_{(2)}| = |(F_{(2)})^*|$. □

Lemma 9.6. *Let $X = \mathbb{T}^2$. Then there exists a distribution $\mu \in \Gamma_B(\mathbb{T}^2)$ such that the characteristic function $\hat{\mu}(y)$ does not vanish and $\mu \notin \Gamma(\mathbb{T}^2)$.*

Proof. Using Lemma 9.2 it is easy to verify that the distribution μ on the two-dimensional torus $\mathbb{T}^2$ constructed in Remark 5.14 has the required property. □

Lemma 9.7. *Let a group X contain no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$, let $\mu \in \Gamma_B(X)$ and assume that the characteristic function $\hat{\mu}(y)$ does not vanish. Then $\mu \in \Gamma(X)$.*

Proof. By Lemma 9.4 the characteristic function $\hat{\mu}(y)$ is represented in the form 9.4 (i). Put $\nu = \mu * \bar{\mu}$. Then it follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}(y) = \exp\{-2\varphi(y)\}$, and hence $\nu \in \Gamma^s(X)$.

Proposition 3.6 yields that $\sigma(\nu) = \widetilde{X}$, where $\widetilde{X}$ is a connected subgroup of the group X. Since μ is a factor of ν, by Proposition 2.2 the distribution μ can be replaced by its shift μ' such that $\sigma(\mu') \subset \widetilde{X}$. Lemma 9.2 implies that $\mu' \in \Gamma_B(X)$. Thus we can assume from the beginning that X is a connected group. There are two possibilities: either the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$ or X contains such a subgroup.

1. Let the group X contain no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Since μ is a factor of ν, by Theorem 4.6 if $\nu \in \Gamma(X)$ then $\mu \in \Gamma(X)$. Hence in this case the lemma is proved.

2. Let the group X contain a subgroup F topologically isomorphic to the circle group $\mathbb{T}$. By Theorem 1.17.1, the subgroup F is a topological direct factor of the group X, i.e., $X = F \times G$, where G is a connected group. Since by the condition of the lemma X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$, the group G contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. By Theorem 1.11.2, $G \cong \mathbb{R}^m \times K$, where $m \geq 0$, and K is a compact connected group. We restrict ourselves to the case $m = 0$. The general case can be considered similarly. For definiteness we will also assume that $\dim K = \aleph_0$. In the case when $\dim K < \infty$ the proof is simplified. By Theorem 1.7.1, $Y = L \times D$, where $L \cong \mathbb{Z}$ and $D \cong K^*$. Applying Theorems 1.6.1 and 1.6.2 we conclude that D is a discrete torsion-free group. By Theorem 1.6.3, $r(D) = \aleph_0$.

By Lemma 9.4 the characteristic function $\hat{\mu}(y)$ can be represented in the form 9.4 (i). The lemma will be proved if we verify that the function $l(y)$ in 9.4 (i) is a character of the group Y, i.e., the function $l(y)$ satisfies the equation

$$l(u + v) = l(u)l(v), \quad u, v \in Y. \tag{9.11}$$

Denote by $y = (n, d), n \in \mathbb{Z}, d \in D$, elements of the group Y. Putting $u = v = y$ in equation 9.4 (ii) and taking into account that $l(0) = 1$, we obtain

$$l(2y) = l^2(y), \quad y \in Y. \tag{9.12}$$

It follows from (9.12) and 9.4 (ii) by induction that $l(ny) = l^n(y)$ for all $y \in Y$, $n \in \mathbb{Z}$. Hence the function $l(n, 0)$, $n \in \mathbb{Z}$, satisfies equation (9.11). By Lemma 9.2 the characteristic function $\hat{\mu}(y)$ satisfies equation 9.2 (i). Consider the restriction of this equation to the subgroup D. By the Bochner theorem $\hat{\mu}(0, d)$, $d \in D$, is the characteristic function of a distribution $\lambda \in \mathrm{M}^1(K)$. Since the function $\hat{\mu}(0, d)$ also satisfies equation 9.2 (i), by Lemma 9.2 we have $\lambda \in \Gamma_B(K)$. Taking into account that the group K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, it follows from case 1 that $\lambda \in \Gamma(K)$. This yields that the function $l(0, d)$, $d \in D$, satisfies equation (9.11).

Consider on the group Y the function

$$a(n, d) = l(n, 0)l(0, d)), \quad (n, d) \in Y.$$

Taking into account what has been said above, the function $a(n, d)$ satisfies equation (9.11). Put

$$b(n, d) = l(n, d)/a(n, d), \quad (n, d) \in Y.$$

It is obvious that the function $b(n,d)$ satisfies equation 9.4 (ii). We will verify that $b(n,d) = 1$ for all $(n,d) \in Y$. Thus the lemma will be proved.

Obviously, $b(n,0) = b(0,d) = 1$ for all $(n,d) \in Y$. Putting $u = (n,0)$, $v = (n,d)$ in equation 9.4 (ii) we get

$$b(2n,d)b(0,-d) = b^2(n,0).$$

We deduce from this that $b(2n,d) = 1$ for all $(n,d) \in Y$. In particular, $b(2n,2d) = 1$ for all $(n,d) \in Y$. Since the function $b(n,d)$ satisfies equation 9.4 (ii), the function $b(n,d)$ satisfies equation (9.12), i.e., $b(2n,2d) = b^2(n,d)$. This implies that $b(n,d) = \pm 1$ for all $(n,d) \in Y$. We will prove that $b(n,d) = 1$ for all $(n,d) \in Y$.

Assume that there exists an element $(n_0,d_0) \in Y$ such that $b(n_0,d_0) = -1$. Choose in D a maximal independent system of elements $d_1,\dots,d_l,\dots$ and let π be the homomorphism $\pi\colon D \mapsto \mathbb{R}^{\aleph_0 *}$ defined by (3.15). Since the group K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Lemma 5.10 there is a subgroup B of D of finite rank m such that $d_0 \in B$ and $\overline{\pi(B)} \cong \mathbb{R}^m$. Let $\tau\colon \overline{\pi(B)} \mapsto \mathbb{R}^m$ be a topological isomorphism. Put $\theta = \tau \circ \pi$. Then $\overline{\theta(B)} = \mathbb{R}^m$.

Denote by $\|t\|$ the norm of a vector $t \in \mathbb{R}^m$. Let $\varepsilon > 0$ be an arbitrary number. Consider the point $s_0 = (\theta d_0)/2 \in \mathbb{R}^m$ and take an element $\tilde{d} \in B$ such that

$$\|s_0 - \theta\tilde{d}\| < \varepsilon/2.$$

Substitute $u = (n_0, d_0 - \tilde{d})$, $v = (0,\tilde{d})$ in equation 9.4 (ii). Taking into account that $b^2(n,d) = 1$ for all $(n,d) \in B$, we obtain

$$b(n_0,d_0)b(n_0,d_0 - 2\tilde{d}) = 1.$$

It follows from this that $b(n_0, d_0 - 2\tilde{d}) = -1$, and moreover

$$\|\theta(d_0 - 2\tilde{d})\| = \|\theta d_0 - 2\theta\tilde{d}\| = 2\|s_0 - \theta\tilde{d}\| < \varepsilon.$$

Set $d' = d_0 - 2\tilde{d}$. Thus we have proved that for any $\varepsilon > 0$ there exists an element $(n_0,d') \in L \times B$ such that $b(n_0,d') = -1$ and $\|\theta d'\| < \varepsilon$.

Since the function $a(n,d)$ is a character of the group Y, the function $b(n,d)\exp\{-\varphi(n,d)\} = \hat{\mu}(y)/a(n,d)$, $(n,d) \in Y$, is positive definite. Denote by $g(n,d)$ its restriction to the subgroup $L \times B$. The function $g(n,d)$ is also positive definite. As follows from the proof of Proposition 3.8 the function $\varphi(n,d)$ on the subgroup $L \times B$ is represented in the form

$$\varphi(n,d) = an^2 + 2n\langle t, \theta d\rangle + \langle A\theta d, \theta d\rangle, \tag{9.13}$$

where $a \ge 0$, $t \in \mathbb{R}^m$, and $A = (\alpha_{i,j})_{i,j=1}^m$ is a symmetric positive semidefinite matrix.

By Bochner's theorem $g(n,d)$ is a characteristic function. Apply inequality 2.7 (g) to the function $g(n,d)$ and put there $u = (n_0,d')$, $v = (n_0,0)$. We get

$$|g(n_0,d') - g(n_0,0)|^2 \le 2(1 - g(0,d')).$$

Taking into account (9.13) we find from this inequality

$$|-\exp\{-an_0^2 - 2n_0\langle t, \theta d'\rangle - \langle A\theta d', \theta d'\rangle\} - \exp\{-an_0^2\}|^2$$
$$= \exp\{-2an_0^2\}|\exp\{-2n_0\langle t, \theta d'\rangle - \langle A\theta d', \theta d'\rangle\} + 1|^2$$
$$\le 2(1 - \exp\{-\langle A\theta d', \theta d'\rangle\}).$$

But this is impossible if the norm $\|\theta d'\|$ is small enough. The contradiction obtained proves that $b(n, d) = 1$ for all $(n, d) \in Y$. We conclude from this that the function $l(n, d) = a(n, d)$ satisfies equation (9.11). □

It should be noted that Lemma 9.6 yields that the statement of Lemma 9.7 is not true if the group X contains a subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Lemma 9.7 implies the following refinement of the group analogue of the Marcinkiewicz theorem (see Theorem 5.11 and Proposition 5.13).

Proposition 9.8. *Let $\mu \in \mathrm{M}^1(X)$ and assume that the characteristic function $\hat\mu(y)$ has the form*

$$\hat\mu(y) = \exp\{\psi(y)\}, \quad \psi(0) = 0, \ y \in Y,$$

where $\psi(y)$ is a polynomial of degree 2. This implies that $\mu \in \Gamma(X)$ if and only if X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$.

Proof. Necessity has been proved in Remark 5.14. Let us prove sufficiency. By Theorem 5.5 the polynomial $\psi(y)$ is represented in the form 5.5 (i). Since $\psi(y)$ is a polynomial of degree 2, representation 5.5 (i) has the form

$$\psi(y) = g_2(y, y) + g_1(y), \quad y \in Y,$$

where $g_2(y_1, y_2)$ is a 2-additive function and $g_1(y)$ an additive function. Substituting the expression

$$\hat\mu(y) = \exp\{g_2(y, y) + g_1(y)\}, \quad y \in Y,$$

into equation 9.2 (i) it is easy to make sure that the characteristic function $\hat\mu(y)$ satisfies this equation. Hence by Lemma 9.2, $\mu \in \Gamma_B(X)$. Since by the condition of the proposition the group X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$, it follows from Lemma 9.7 that $\mu \in \Gamma(X)$. □

We will prove now the main theorem of this section.

Theorem 9.9. *Equality (9.2) holds for a group X if and only if the connected component of zero of X contains no more than one element of order 2.*

Proof. Necessity. Essentially the proof is the same as the proof of item (II) in Theorem 7.10. Assume that the connected component of zero of X contains more than one element of order 2. Then by Lemma 9.5 there exists a compact Corwin subgroup K of the group X such that the factor group X/K contains a subgroup F topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Therefore we can consider the distribution $\mu \in \Gamma_B(\mathbb{T}^2)$, $\mu \notin \Gamma(\mathbb{T}^2)$, constructed in Lemma 9.6 as a distribution on the factor group X/K. This distribution we will also denote by μ. We may suppose that the characteristic function $\hat{\mu}(y)$ is defined on the annihilator $A(Y, K)$ because by Theorem 1.9.2, $(X/K)^* \cong A(Y, K)$. Consider on the group Y the function

$$h(y) = \begin{cases} \hat{\mu}(y) & \text{if } y \in A(Y, K), \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

Since $A(Y, K)$ is a subgroup and $\hat{\mu}(y)$ is a positive definite function, by Proposition 2.12, $h(y)$ is also a positive definite function. By Theorem 1.9.4 the annihilator $A(Y, K)$ is an open subgroup because K is a compact subgroup. Hence the function $h(y)$ is continuous. By the Bochner theorem there exists a distribution $\lambda \in \mathrm{M}^1(X)$ such that $\hat{\lambda}(y) = h(y)$.

We will check that $\lambda \in \Gamma_B(X)$. In view of Lemma 9.2 it suffices to show that the characteristic function $\hat{\lambda}(y)$ satisfies equation 9.2 (i). This verification is the same as the corresponding verification in the proof of statement (II) in Theorem 7.10.

Since $\hat{\mu}(m, n) \neq 0$ for all $(m, n) \in \mathbb{Z}^2$ and $\mu \notin \Gamma(\mathbb{T}^2)$, it follows from this that $\lambda \notin \Gamma(X) * I_B(X)$. The necessity is proved.

Sufficiency. Let $\mu \in \Gamma_B(X)$. Consider the set $N = \{y \in Y : \hat{\mu}(y) \neq 0\}$. By Lemma 9.2 the characteristic function $\hat{\mu}(y)$ satisfies equation 9.2 (i). This yields that N is a subgroup of Y. Obviously, N is an open subgroup. Put $K = A(X, N)$. Then by Theorem 1.9.4, K is a compact group. Substituting $u = v = y$ into equation 9.2 (i) we get

$$\hat{\mu}(2y) = \hat{\mu}^2(y)|\hat{\mu}^2(y)|, \quad y \in Y.$$

It follows from this that if $2y \in N$ then $y \in N$. Applying Lemma 7.2 we obtain that K is a Corwin group. We also note that $N = A(Y, K)$ by Theorem 1.9.1. We deduce from Theorem 1.9.2 that $(X/K)^* \cong A(Y, K)$. Hence by Lemma 9.2 and Corollary 2.11 the restriction of the characteristic function $\hat{\mu}(y)$ to N is the characteristic function of a distribution $\lambda \in \Gamma_B(X/K)$. Taking into account the condition of the theorem and Lemma 9.5 we conclude that the factor group X/K contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Applying Lemma 9.7 we get that $\lambda \in \Gamma(X/K)$. Taking into account Theorem 1.9.2, we obtain for the characteristic function of the distribution λ the representation

$$\hat{\lambda}(y) = (x, y) \exp\{-\varphi(y)\}, \quad x \in X, \ y \in N, \tag{9.14}$$

where $\varphi(y)$ is a continuous nonnegative function on N satisfying equation 2.16 (ii). By Lemma 3.18 the function $\varphi(y)$ can be extended from the subgroup N to Y retaining its properties. Denote by $\tilde{\varphi}(y)$ the extended function. Let γ be a Gaussian distribution on X with the characteristic function

$$\hat{\gamma}(y) = (x, y) \exp\{-\tilde{\varphi}(y)\}, \quad y \in Y. \tag{9.15}$$

It follows from 2.14 (i) that the characteristic function of the Haar distribution of the subgroup K is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in N, \\ 0 & \text{if } y \notin N. \end{cases} \tag{9.16}$$

Combining (9.14)–(9.16) we get $\hat{\mu}(y) = \hat{\gamma}(y)\widehat{m}_K(y)$. Applying 2.7 (b) and 2.7 (c) this yields that $\mu = \gamma * m_K$, i.e., $\mu \in \Gamma(X) * I_B(X)$. □

Below we give another proof of the sufficiency in Theorem 9.9. This proof is not based on Lemma 9.7. We recall that the proof of Lemma 9.7 is based on Theorem 4.6 (the group analogue of the Cramér theorem)(compare with Remark 7.12). We need two lemmas.

Lemma 9.10. *Let $X = \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. Then $X_{(2)} \subset c_X$.*

Proof. Applying Theorem 1.7.1 we get $Y = L \times M$, where $L \cong \mathbb{R}^m$, $M \cong K^*$. By Theorem 1.6.1 the group M is discrete. Since $K^{(2)} = K$, by Theorem 1.9.5, $M_{(2)} = \{0\}$, i.e., the group M contains no elements of order 2. We conclude from this that the subgroup b_M consists of zero and all elements of odd order of the group M. Hence $(b_M)^{(2)} = b_M$. This implies that $b_M \subset Y^{(2)}$. By Theorem 1.9.3, $c_X = A(X, b_Y)$. It is obvious that $b_Y = b_M$. Therefore $A(X, b_M) = A(X, b_Y)$. By Theorem 1.9.5, $A(X, Y^{(2)}) = X_{(2)}$. From what has been said it follows that $c_X = A(X, b_Y) = A(X, b_M) \supset A(X, Y^{(2)}) = X_{(2)}$. □

Lemma 9.11. *Let a group X contain no more than one element of order 2, let $\mu \in \Gamma_B(X)$ and assume that the characteristic function $\hat{\mu}(y)$ does not vanish. Then $\mu \in \Gamma(X)$.*

Proof. By Lemma 9.4 the characteristic function $\hat{\mu}(y)$ can be represented in the form 9.4 (i). The lemma will be proved if we verify that the function $l(y)$ in 9.4 (i) is a character of the group Y, i.e., $l(y)$ satisfies equation (9.11).

Swap u and v in equation 9.4 (ii). Multiplying the resulting equation by 9.4 (ii) we find

$$l^2(u+v) = l^2(u)l^2(v), \quad u, v \in Y. \tag{9.17}$$

Substituting $u = v = y$ into 9.4 (ii) we obtain $l(2y) = l^2(y)$, $y \in Y$. It follows from this and (9.17) that

$$l(2u+2v) = l(2u)l(2v), \quad u, v \in Y. \tag{9.18}$$

Equality (9.18) yields that the function $l(y)$ is a character of the subgroup $\overline{Y^{(2)}}$.

If the group X contains no elements of order 2, then by Theorem 1.9.5, $Y = \overline{Y^{(2)}}$ and the function $l(y)$ is a character of the group Y.

Let the group X contain an element of order 2. We will show that in this case the function $l(y)$ is also a character of the group Y. By Theorem 1.9.2 we have

$l(y) = (x_0, y)$, $x_0 \in X$, $y \in \overline{Y^{(2)}}$. Put $m(y) = l(y)/(x_0, y)$. The function $m(y)$ also satisfies equation 9.4 (ii) and $m(y) = 1$ for all $y \in \overline{Y^{(2)}}$. Since $m(2y) = m^2(y)$ for all $y \in Y$, we get

$$m(y) = \pm 1, \quad y \in Y. \tag{9.19}$$

Equation 9.4 (ii) for the function $m(y)$ becomes equation $m(u+v)m(u-v) = 1$, $u, v \in Y$. Substituting here $u + v$ for u we infer that $m(u+2v)m(u) = 1$, $u, v \in Y$. Hence $m(u+2v) = m(u)$, $u, v \in Y$, i.e., the function $m(y)$ is invariant with respect to shifts by elements of $\overline{Y^{(2)}}$. Consider the factor group $Y/\overline{Y^{(2)}}$. Theorems 1.9.2 and 1.9.5 yield that $(Y/\overline{Y^{(2)}})^* \cong A(X, \overline{Y^{(2)}}) = X_{(2)} \cong \mathbb{Z}(2)$. Therefore the factor group $Y/\overline{Y^{(2)}}$ consists of two cosets, i.e., $Y/\overline{Y^{(2)}} = \overline{Y^{(2)}} \cup (y_0 + \overline{Y^{(2)}})$. The function $m(y)$ takes a constant value on each coset. It follows from (9.19) that we have two possibilities: either $m(y) = 1$ for all $y \in Y$ or

$$m(y) = \begin{cases} 1 & \text{if } y \in \overline{Y^{(2)}}, \\ -1 & \text{if } y \in y_0 + \overline{Y^{(2)}}. \end{cases}$$

Obviously, in both cases the function $m(y)$ satisfies equation (9.11) on the group Y. Therefore $m(y)$ is a character, and hence $l(y)$ is also a character. □

It is obvious that Lemma 9.11 follows from Lemma 9.7. However, as opposed to Lemma 9.7 the proof of Lemma 9.11 does not use the group analogue of the Cramér theorem (Theorem 4.6).

The proof of Lemma 9.11 implies directly two corollaries.

Corollary 9.12. *Assume that a function $l(y)$ on a group Y is continuous, satisfies equation* 9.4 (ii) *and the conditions $l(-y) = \overline{l(y)}$, $|l(y)| = 1$, $y \in Y$, $l(0) = 1$. Then*

$$l(y) = m(y)(x_0, y),$$

where $m(y)$ is a $\overline{Y^{(2)}}$-invariant continuous function satisfying equation 9.4 (ii)*, the conditions $m(-y) = m(y)$ for all $y \in Y$, $m(0) = 1$, and taking values ± 1, $x_0 \in X$.*

Corollary 9.13. *Let a group X contain no more than one element of order* 2 *and assume that a function $l(y)$ on Y is continuous, satisfies equation* 9.4 (ii)*, and the conditions $l(-y) = \overline{l(y)}$, $|l(y)| = 1$, $y \in Y$, $l(0) = 1$. Then $l(y)$ is a character of the group Y.*

9.14 Second proof of sufficiency in Theorem 9.9. It follows from Definition 9.1 and Lemma 7.5 that we can assume without loss of generality that $X = \mathbb{R}^m \times K$, where $m \geq 0$ and K is a compact Corwin group. By Lemma 9.10 for such groups the inclusion $X_{(2)} \subset c_X$ holds. Since by the condition of the theorem the subgroup c_X contains no more than one element of order 2, the group X also contains no more than one element of order 2. Put $N = \{y \in Y : \hat{\mu}(y) \neq 0\}$, $G = A(X, N)$. By Theorem 1.9.2, $N^* \cong X/G$. As has been noted in the proof of sufficiency in Theorem 9.9, G is a Corwin group. We will check that the factor group X/G contains no more than one element of order 2. Indeed, let $2[x_1] = 2[x_2] = [0]$, $[x_1] \neq [x_2]$, $[x_1], [x_2] \in X/G$. Then $2x_j \in G$. Since $G = G^{(2)}$, we have $2x_j = 2g_j$, $g_j \in G$. Hence $x'_j = x_j - g_j$ are elements of order 2. Taking into account that $[x'_j] = [x_j]$ and $[x_1] \neq [x_2]$, we obtain $x'_1 \neq x'_2$. But this contradicts the condition of the theorem.

Consider the restriction of equation 9.2 (i) to N. Applying Lemma 9.11 to the factor group X/G we obtain that this restriction is the characteristic function of a Gaussian distribution on X/G. This immediately yields that $\mu \in \Gamma(X) * I_B(X)$. □

Remark 9.15. Note that if γ is an infinitely divisible distribution on a group X and $\gamma \in \Gamma_B(X)$, then $\gamma \in \Gamma(X)$. To show this we reason as in the proof of sufficiency in Theorem 9.9. Instead of Lemma 9.7 we use the following statement: if γ is an infinitely divisible distribution, $\gamma \in \Gamma_B(X)$, and the characteristic function $\hat{\gamma}(y)$ does not vanish, then $\gamma \in \Gamma(X)$. To prove this statement consider the distribution $\lambda = \gamma * \bar{\gamma}$. Combining Lemma 9.4, 2.7 (c), and 2.7 (d), we get $\lambda \in \Gamma(X)$. Taking into account that γ is a factor of λ, by Remark 4.8, $\gamma \in \Gamma(X)$.

Remark 9.16. It follows from Definition 3.1 that any Gaussian distribution can be represented as a convolution of a symmetric Gaussian distribution and a degenerate distribution. It turns out that the analogous statement is also true for the Gaussian distributions in the sense of Bernstein. Indeed, let $\mu \in \Gamma_B(X)$. By Lemma 9.4 the characteristic function $\hat{\mu}(y)$ can be represented in the form 9.4 (i). As has been shown in the proof of Lemma 9.11 there exists a character (x_0, y) such that $\hat{\mu}(y)/(x_0, y)$ is a real-valued function. This yields the desired statement.

Chapter IV
The Skitovich–Darmois theorem for locally compact Abelian groups (the characteristic functions of random variables do not vanish)

Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, α_j, β_j be nonzero real numbers. According to the Skitovich–Darmois theorem if the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent, then all random variables ξ_j are Gaussian. This theorem was generalized by Ghurye and Olkin to the case when, instead of random variables, random vectors ξ_j in the space $\mathbb{R}^m$ are considered, and coefficients of the linear forms L_1 and L_2 are nonsingular $(m \times m)$-matrices. They proved that the independence of the linear forms L_1 and L_2 implies that the random vectors ξ_j are Gaussian.

Let X be a second countable locally compact Abelian group, ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. This chapter is devoted to some group analogues of the Skitovich–Darmois theorem. We will assume that the characteristic functions of the random variables ξ_j do not vanish. Let α_j, β_j be topological automorphisms of X. Suppose that X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. We prove that in this case if the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent, then all distributions μ_j are Gaussian. This statement fails if X contains a subgroup topologically isomorphic to $\mathbb{T}$. A natural problem arises for such groups: to describe all possible distributions μ_j of independent random variables ξ_j assuming that the linear forms L_1 and L_2 are independent. We solve this problem for two independent random variables taking values in the groups $\mathbb{T}^2$, $\mathbb{R} \times \mathbb{T}$, and $\Sigma_{\boldsymbol{a}} \times \mathbb{T}$.

10 Locally compact Abelian groups for which the Skitovich–Darmois theorem holds

Let X be a second countable locally compact Abelian group, Y be its character group, $\mathrm{Aut}(X)$ be the group of topological automorphisms of the group X. In this section we describe groups X that have the following property: if ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions and $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, then the independence of the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ implies that all distributions $\mu_j \in \Gamma(X)$. We also describe groups X that have the following property: there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ such that if ξ_j are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the independence of the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ implies that all distributions $\mu_j \in \Gamma(X)$.

Lemma 10.1. *Let α_j, β_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of a group X. Let ξ_j be independent random variables with values in X and distributions μ_j. The linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation*

$$\text{(i)} \quad \prod_{i=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \prod_{i=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u) \prod_{i=1}^{n} \hat{\mu}_j(\tilde{\beta}_j v), \quad u, v \in Y.$$

Proof. We note that the linear forms L_1 and L_2 are independent if and only if the equality

$$\begin{aligned} &\mathbf{E}[(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, u)(\beta_1\xi_1 + \cdots + \beta_n\xi_n, v)] \\ &\qquad = \mathbf{E}[(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, u)]\mathbf{E}[(\beta_1\xi_1 + \cdots + \beta_n\xi_n, v)] \end{aligned} \tag{10.1}$$

holds for all $u, v \in Y$. Taking into account that the random variables ξ_j are independent and $\hat{\mu}_j(y) = \mathbf{E}[(\xi_j, y)]$, we transform the left-hand side of equality (10.1) as follows:

$$\begin{aligned} &\mathbf{E}[(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, u)(\beta_1\xi_1 + \cdots + \beta_n\xi_n, v)] \\ &\qquad = \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u + \tilde{\beta}_j v)\Big] \\ &\qquad = \prod_{j=1}^{n} \mathbf{E}[(\xi_j, \tilde{\alpha}_j u + \tilde{\beta}_j v)] \\ &\qquad = \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v), \quad u, v \in Y. \end{aligned}$$

We transform similarly the right-hand side of equality (10.1):

$$\begin{aligned} &\mathbf{E}[(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, u)]\mathbf{E}[(\beta_1\xi_1 + \cdots + \beta_n\xi_n, v)] \\ &\qquad = \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u)\Big]\mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\beta}_j v)\Big] \\ &\qquad = \prod_{j=1}^{n} \mathbf{E}[(\xi_j, \tilde{\alpha}_j u)] \prod_{j=1}^{n} \mathbf{E}[(\xi_j, \tilde{\beta}_j v)] \\ &\qquad = \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u) \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\beta}_j v), \quad u, v \in Y. \end{aligned}$$

□

Equation (i) is called the *Skitovich–Darmois functional equation*.

Corollary 10.2. *Let α_j, β_j, $j = 1, 2, \dots, n$, $n \ge 2$, be topological automorphisms of a group X. Let ξ_j be independent random variables with values in X and distributions μ_j. Assume that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent. If ξ_j' are shifts of the random variables ξ_j, then the linear forms $L_1' = \alpha_1\xi_1' + \cdots + \alpha_n\xi_n'$ and $L_2' = \beta_1\xi_1' + \cdots + \beta_n\xi_n'$ are also independent.*

Theorem 10.3. *Let X be a group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let α_j, β_j, $j = 1, 2, \dots, n$, $n \ge 2$, be topological automorphisms of X. Let ξ_j be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. Then if the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, all $\mu_j \in \Gamma(X)$.*

Proof. First note that if μ is the distribution of a random variable ξ taking values in a group X and $\alpha \in \mathrm{Aut}(X)$, then by Proposition 2.10 the characteristic function of the random variable $\alpha\xi$ is equal to $\hat\mu(\tilde\alpha y)$. Taking into account Definition 3.1, it follows from this that $\mu \in \Gamma(X)$ if and only if $\alpha(\mu) \in \Gamma(X)$. Therefore we can put $\zeta_j = \alpha_j\xi_j$ and reduce the proof of the theorem to the case when $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, $\delta_j \in \mathrm{Aut}(X)$. By Lemma 10.1 it follows from the independence of L_1 and L_2 that the characteristic functions $\hat\mu_j(y)$ satisfy equation 10.1 (i) which takes the form

$$\prod_{j=1}^{n} \hat\mu_j(u + \tilde\delta_j v) = \prod_{j=1}^{n} \hat\mu_j(u) \prod_{j=1}^{n} \hat\mu_j(\tilde\delta_j v), \quad u, v \in Y. \tag{10.2}$$

Put $\nu_j = \mu_j * \bar\mu_j$. We deduce from 2.7 (c) and 2.7 (d) that $\hat\nu_j(y) = |\hat\mu_j(y)|^2 > 0$, $y \in Y$. Obviously, the characteristic functions $\hat\nu_j(y)$ also satisfy equation (10.2). If we prove that $\nu_j \in \Gamma(X)$, then applying Theorem 4.6 we get that $\mu_j \in \Gamma(X)$. Therefore we can assume from the beginning that the characteristic functions $\hat\mu_j(y) > 0$, $y \in Y$, $j = 1, 2, \dots, n$, $n \ge 2$.

Put $\varphi_j(y) = -\ln \hat\mu_j(y)$. It follows from (10.2) that the functions $\varphi_j(y)$ satisfy the equation

$$\sum_{j=1}^{n} \varphi_j(u + \tilde\delta_j v) = P(u) + Q(v), \quad u, v \in Y, \tag{10.3}$$

where

$$P(u) = \sum_{j=1}^{n} \varphi_j(u), \quad Q(v) = \sum_{j=1}^{n} \varphi_j(\tilde\delta_j v). \tag{10.4}$$

We use the finite difference method to solve equation (10.3). Let h_1 be an arbitrary element of the group Y. Set $k_1 = -\tilde\delta_n^{-1} h_1$, then $h_1 + \tilde\delta_n k_1 = 0$. Substitute $u + h_1$ for u and $v + k_1$ for v in equation (10.3). Subtracting equation (10.3) from the resulting equation we obtain

$$\sum_{j=1}^{n-1} \Delta_{l_{1j}} \varphi_j(u + \tilde\delta_j v) = \Delta_{h_1} P(u) + \Delta_{k_1} Q(v), \quad u, v \in Y, \tag{10.5}$$

where $l_{1j} = h_1 + \tilde{\delta}_j k_1 = (\tilde{\delta}_j - \tilde{\delta}_n)k_1$, $j = 1, 2, \dots, n-1$. Let h_2 be an arbitrary element of the group Y. Put $k_2 = -\tilde{\delta}_{n-1}^{-1} h_2$, then $h_2 + \tilde{\delta}_{n-1}k_2 = 0$. Substitute $u + h_2$ for u and $v + k_2$ for v in equation (10.5). Subtracting equation (10.5) from the resulting equation we obtain

$$\sum_{j=1}^{n-2} \Delta_{l_{2j}} \Delta_{l_{1j}} \varphi_j(u + \tilde{\delta}_j v) = \Delta_{h_2}\Delta_{h_1} P(u) + \Delta_{k_2}\Delta_{k_1} Q(v), \quad u, v \in Y, \tag{10.6}$$

where $l_{2j} = h_2 + \tilde{\delta}_j k_2 = (\tilde{\delta}_j - \tilde{\delta}_{n-1})k_2$, $j = 1, 2, \dots, n-2$. Arguing as above we obtain the equation

$$\begin{aligned} &\Delta_{l_{n-1,1}} \Delta_{l_{n-2,1}} \dots \Delta_{l_{11}} \varphi_1(u + \tilde{\delta}_1 v) \\ &\quad = \Delta_{h_{n-1}} \Delta_{h_{n-2}} \dots \Delta_{h_1} P(u) + \Delta_{k_{n-1}} \Delta_{k_{n-2}} \dots \Delta_{k_1} Q(v), \quad u, v \in Y, \end{aligned} \tag{10.7}$$

where h_m is an arbitrary element of Y, $k_m = -\tilde{\delta}_{n-m+1}^{-1} h_m$, $m = 1, 2, \dots, n-1$, $l_{mj} = h_m + \tilde{\delta}_j k_m = (\tilde{\delta}_j - \tilde{\delta}_{n-m+1})k_m$, $j = 1, 2, \dots, n-m$. Let h_n be an arbitrary element of the group Y. Put $k_n = -\tilde{\delta}_1^{-1} h_n$, then $h_n + \tilde{\delta}_1 k_n = 0$. Substitute $u + h_n$ for u and $v + k_n$ for v in equation (10.7). Subtracting equation (10.7) from the resulting equation we obtain

$$\Delta_{h_n} \Delta_{h_{n-1}} \dots \Delta_{h_1} P(u) + \Delta_{k_n} \Delta_{k_{n-1}} \dots \Delta_{k_1} Q(v) = 0, \quad u, v \in Y. \tag{10.8}$$

Let h be an arbitrary element of the group Y. Substitute $u + h$ for u in equation (10.8) and subtract equation (10.8) from the resulting equation. We get

$$\Delta_h \Delta_{h_n} \Delta_{h_{n-1}} \dots \Delta_{h_1} P(u) = 0, \quad u \in Y. \tag{10.9}$$

We note that u, h, and h_m, $m = 1, 2, \dots, n$ are arbitrary elements of the group Y. We can put $h_1 = \dots = h_n = h$ in (10.9). We have

$$\Delta_h^{n+1} P(u) = 0, \quad u, h \in Y. \tag{10.10}$$

Thus $P(y)$ is a continuous polynomial on Y. Set $\gamma = \mu_1 * \dots * \mu_n$. Then 2.7 (c) yields that

$$\hat{\gamma}(y) = \prod_{j=1}^{n} \hat{\mu}_j(y), \quad y \in Y.$$

In view of (10.4) we have

$$\hat{\gamma}(y) = e^{-P(y)}.$$

Since the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$ and $P(y)$ is a continuous polynomial, by Theorem 5.11, $\gamma \in \Gamma(X)$. Applying Theorem 4.6 we conclude that all $\mu_j \in \Gamma(X)$. □

Remark 10.4. Theorem 10.3 is not valid if a group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$. This immediately follows from Lemma 7.8 (compare below with Proposition 10.16).

Remark 10.5. Let $\alpha_j, \beta_j, j = 1, 2, \ldots, n, n \geq 2$, be topological automorphisms of a group X. Let ξ_j be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. Then it follows from the independence of the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ that the random variables ξ_j can be replaced by their shifts ξ'_j in such a way that their distributions μ'_j are supported in c_X.

To prove this, we note that because c_X is a characteristic subgroup, we can assume that $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, where $\delta_j \in \mathrm{Aut}(X)$. Put $\nu_j = \mu_j * \bar{\mu}_j, \varphi_j(y) = -\ln \hat{\nu}_j(y)$, $P(y) = \sum_{j=1}^n \varphi_j(y)$. As follows from the proof of Theorem 10.3, the function $P(y)$ satisfies equation (10.10). By Proposition 5.7 the function $P(y) = 0$ for all $y \in b_Y$. This yields that all characteristic functions $\hat{\nu}_j(y) = 1$ for all $y \in b_Y$. By Proposition 2.13 the inclusions $\sigma(\nu_j) \subset A(X, b_Y)$, $j = 1, 2, \ldots, n$, hold. By Theorem 1.9.3, $c_X = A(X, b_Y)$. Hence all supports $\sigma(\nu_j) \subset c_X$.

Now the desired statement follows from Proposition 2.2. We note that c_X is a characteristic subgroup and in view of Corollary 10.2 the linear forms $L'_1 = \alpha_1\xi'_1 + \cdots + \alpha_n\xi'_n$ and $L'_2 = \beta_1\xi'_1 + \cdots + \beta_n\xi'_n$ are also independent.

10.6. It should be noted that one can consider another group analogue of the Skitovich–Darmois theorem. Namely, we will assume that coefficients of linear forms L_1 and L_2 are integers. We need the following definition. A set of integers $\{a_j\}$ is said to be *admissible for a group* X if $X^{(a_j)} \neq \{0\}$ for all j. Let ξ_j, $j = 1, 2, \ldots, n, n \geq 2$, be random variables taking values in the group X. The admissibility of a set $\{a_j\}_{j=1}^n$ when considering the linear form $L = a_1\xi_1 + \cdots + a_n\xi_n$ is a group analogue of the condition $a_j \neq 0$, $j = 1, 2, \ldots, n$, for the case of $X = \mathbb{R}$. The following result holds.

Proposition 10.7. *Let $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ be admissible sets of integers for a group X. Let ξ_j be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. The independence of the linear forms $L_1 = a_1\xi_1 + \cdots + a_n\xi_n$ and $L_2 = b_1\xi_1 + \cdots + b_n\xi_n$ yields that all $\mu_j \in \Gamma(X)$ if and only if either X is a torsion-free group or $X^{(p)} = \{0\}$, where p is a prime number* (*in the last case all μ_j are degenerate distributions*).

Proof. Note that if $a \in \mathbb{Z}$, then in view of 1.13 (d) $\tilde{f}_a = f_a$. Reasoning as in the proof of Lemma 10.1, we obtain that the linear forms L_1 and L_2 are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\prod_{j=1}^n \hat{\mu}_j(a_j u + b_j v) = \prod_{j=1}^n \hat{\mu}_j(a_j u) \prod_{j=1}^n \hat{\mu}_j(b_j v), \quad u, v \in Y. \tag{10.11}$$

Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.11).

Assume first that X is a torsion-free group. Then the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Therefore, if we prove that $\nu_j \in \Gamma(X)$, then applying Theorem 4.6 we obtain that $\mu_j \in \Gamma(X)$. Hence we can assume from the beginning that $\hat{\mu}_j(y) > 0$, $y \in Y$, $j = 1, 2, \ldots, n$, $n \geq 2$.

Set $\varphi_j(y) = -\ln \hat{\mu}_j(y)$. We conclude from (10.11) that

$$\sum_{j=1}^{n} \varphi_j(a_j u + b_j v) = P(u) + Q(v), \quad u, v \in Y, \tag{10.12}$$

where

$$P(u) = \sum_{j=1}^{n} \varphi_j(a_j u), \quad Q(v) = \sum_{j=1}^{n} \varphi_j(b_j v). \tag{10.13}$$

We use again the finite difference method to solve equation (10.12). Let h_n be an arbitrary element of the group Y. Substitute $u + b_n h_n$ for u and $v - a_n h_n$ for v in equation (10.12). Subtracting equation (10.12) from the resulting equation we obtain

$$\sum_{j=1}^{n-1} \Delta_{l_{n,j}} \varphi_j(a_j u + b_j v) = \Delta_{b_n h_n} P(u) + \Delta_{-a_n h_n} Q(v), \quad u, v \in Y, \tag{10.14}$$

where $l_{n,j} = (a_j b_n - b_j a_n) h_n$, $j = 1, 2, \ldots, n-1$. Let h_{n-1} be an arbitrary element of the group Y. Substitute $u + b_{n-1} h_{n-1}$ for u and $v - a_{n-1} h_{n-1}$ for v in equation (10.14). Subtracting equation (10.14) from the resulting equation we obtain

$$\begin{aligned} &\sum_{j=1}^{n-2} \Delta_{l_{n-1,j}} \Delta_{l_{n,j}} \varphi_j(a_j u + b_j v) \\ &\quad = \Delta_{b_{n-1} h_{n-1}} \Delta_{b_n h_n} P(u) + \Delta_{-a_{n-1} h_{n-1}} \Delta_{-a_n h_n} Q(v), \quad u, v \in Y, \end{aligned} \tag{10.15}$$

where $l_{n-1,j} = (a_j b_{n-1} - b_j a_{n-1}) h_{n-1}$, $j = 1, 2, \ldots, n-2$. Arguing as above we get the equation

$$\begin{aligned} &\Delta_{l_{2,1}} \Delta_{l_{3,1}} \ldots \Delta_{l_{n,1}} \varphi_1(a_1 u + b_1 v) \\ &\quad = \Delta_{b_2 h_2} \Delta_{b_3 h_3} \ldots \Delta_{b_n h_n} P(u) + \Delta_{-a_2 h_2} \Delta_{-a_3 h_3} \ldots \Delta_{-a_n h_n} Q(v), \quad u, v \in Y, \end{aligned} \tag{10.16}$$

where h_m is an arbitrary element of the group Y, $l_{m,1} = (a_1 b_m - b_1 a_m) h_m$, $m = 2, 3, \ldots, n$.

Let h_1 be an arbitrary element of the group Y. Substitute $u + b_1 h_1$ for u and $v - a_1 h_1$ for v in equation (10.16). Subtracting equation (10.16) from the resulting equation we obtain

$$\begin{aligned} &\Delta_{b_1 h_1} \Delta_{b_2 h_2} \Delta_{b_3 h_3} \ldots \Delta_{b_n h_n} P(u) \\ &\quad + \Delta_{-a_1 h_1} \Delta_{-a_2 h_2} \Delta_{-a_3 h_3} \ldots \Delta_{-a_n h_n} Q(v) = 0, \quad u, v \in Y. \end{aligned} \tag{10.17}$$

Let h be an arbitrary element of the group Y. Substitute $u + h$ for u in equation (10.17) and subtract equation (10.17) from the resulting equation. We get

$$\Delta_h \Delta_{b_1 h_1} \Delta_{b_2 h_2} \Delta_{b_3 h_3} \ldots \Delta_{b_n h_n} P(u) = 0, \quad u \in Y. \tag{10.18}$$

We assumed that X is a torsion-free group, i.e., $X_{(m)} = \{0\}$ for all $m \in \mathbb{Z}, m \neq 0$. This implies by Theorem 1.9.5 that $\overline{Y^{(m)}} = Y$ for all $m \in \mathbb{Z}$, $m \neq 0$. In particular, $\overline{Y^{(b_j)}} = Y$, $j = 1, 2, \ldots, n$. Hence taking into account that h_j are arbitrary elements of the group Y, (10.18) yields that the function $P(y)$ satisfies the equation

$$\Delta_h^{n+1} P(u) = 0, \quad u, h \in Y.$$

So $P(y)$ is a continuous polynomial on the group Y. Consider the distribution $\gamma = f_{a_1}(\mu_1) * \cdots * f_{a_n}(\mu_n)$. We conclude from Proposition 2.10 and 1.13 (d) that $\widehat{f_{a_j}(\mu_j)}(y) = \hat{\mu}(a_j y)$, $j = 1, 2, \ldots, n$. Then 2.7 (c) implies that

$$\hat{\gamma}(y) = \prod_{j=1}^{n} \hat{\mu}_j(a_j y), \quad y \in Y.$$

Taking into account (10.13), we have

$$\hat{\gamma}(y) = e^{-P(y)}.$$

It follows from the condition of the theorem that the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Hence Theorem 5.11 implies that $\gamma \in \Gamma(X)$. Then by Theorem 4.6 we obtain that all $f_{a_j}(\mu_j) \in \Gamma(X)$. This implies that every function $\varphi_j(y)$ satisfies equation 2.16 (ii) on the set $Y^{(a_j)}$. Since $\overline{Y^{(a_j)}} = Y$, $j = 1, 2, \ldots, n$, every function $\varphi_j(y)$ satisfies equation 2.16 (ii) on the group Y. But this means that all $\mu_j \in \Gamma(X)$.

If $X^{(p)} = \{0\}$, where p is a prime number, then it follows from admissibility of the sets $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ for the group X that $f_{a_j}, f_{b_j} \in \mathrm{Aut}(X)$, $j = 1, 2, \ldots, n$. Taking into account that $c_X = \{0\}$, Remark 10.5 yields that all $\mu_j \in D(X)$.

Assume now that a group X contains an element x_0 of order p, where p is a prime number, but $X^{(p)} \neq \{0\}$. Let M be the subgroup of X generated by the element x_0. We have $M \cong \mathbb{Z}(p)$. Let ξ_1 and ξ_2 be independent random variables with values in the subgroup M and nondegenerate distributions μ_j with non-vanishing characteristic functions. Consider the linear forms $L_1 = p\xi_1 - \xi_2$ and $L_2 = \xi_1 + p\xi_2$. It is easy to see that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (10.11). Hence the linear forms L_1 and L_2 are independent. Obviously, the sets $\{p, -1\}$ and $\{1, p\}$ are admissible for the group X. In view of Proposition 3.6, $\Gamma(M) = D(M)$. Hence $\mu_j \notin \Gamma(X)$. □

Remark 10.8. Let $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ be admissible sets of integers for a group X and let ξ_j be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. Denote by k and l the least common multiple of the numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ respectively. Let p_j be the greatest common divisor of the numbers $\{a_j l, b_j k\}$ and m be the least common multiple of the numbers $\{p_1, \ldots, p_n\}$. Then the independence of the linear forms $L_1 = a_1\xi_1 + \cdots + a_n\xi_n$ and $L_2 = b_1\xi_1 + \cdots + b_n\xi_n$ yields that the random variables ξ_j can be replaced by their shifts ξ_j' in such a way that their distributions μ_j' are supported in the subgroup $\{x \in X : mx \in c_X\}$.

To prove this, put $\nu_j = \mu_j * \bar{\mu}_j$, $\varphi_j(y) = -\ln \hat{\nu}_j(y)$, $P(y) = \sum_{j=1}^{n} \varphi_j(a_j y)$. Arguing as in the proof of Proposition 10.7 we obtain that the function $P(y)$ satisfies equation (10.18). It follows from this that the function $P(y)$ is a polynomial on the subgroup $Y^{(l)}$. By Proposition 5.7, $P(y) = 0$ for $y \in b_{\overline{Y^{(l)}}}$, and hence $\hat{\nu}_j(y) = 1$, $y \in b_{\overline{Y^{(a_j l)}}}$, $j = 1, 2, \dots, n$. By Proposition 2.13, the inclusions $\sigma(\nu_j) \subset A(X, b_{\overline{Y^{(a_j l)}}})$ hold. Put $A_j = A(X, b_{\overline{Y^{(a_j l)}}})$. Changing places the forms L_1 and L_2 and reasoning as above we find that $\sigma(\nu_j) \subset B_j$, where $B_j = A(X, b_{\overline{Y^{(b_j k)}}})$. Put $C_j = A_j \cap B_j$. So we have

$$\sigma(\nu_j) \subset C_j. \qquad (10.19)$$

It follows easily from Theorem 1.9.3 that $A_j = \{x \in X : a_j l x \in c_X\}$ and $B_j = \{x \in X : b_j k x \in c_X\}$. Hence, $C_j = \{x \in X : p_j x \in c_X\}$. Put $C = \{x \in X : m x \in c_X\}$. Obviously,

$$C_j \subset C, \quad j = 1, 2, \dots, n, \qquad (10.20)$$

and C is the least subgroup containing all C_j. Applying Proposition 2.2 we obtain from (10.19) and (10.20) the desired statement.

It should be noted that by Corollary 10.2, the linear forms $L_1 = a_1\xi_1' + \dots + a_n\xi_n'$ and $L_2 = b_1\xi_1' + \dots + b_n\xi_n'$ are independent.

In what follows we need two lemmas.

Lemma 10.9. *Let Y be an arbitrary Abelian group, ε be an automorphism of the group Y. Assume that functions $\varphi_j(y)$ satisfy the equation*

$$\text{(i)} \quad \varphi_1(u + v) + \varphi_2(u + \varepsilon v) = P(u) + Q(v), \quad u, v \in Y.$$

Then each of the functions $\varphi_j(y)$ satisfies the equation

$$\text{(ii)} \quad \Delta_{(I-\varepsilon)k}\Delta_h^2 \varphi_j(u) = 0, \quad k, h, u \in Y.$$

Proof. Let k be an arbitrary element of the group Y. Put $h = -\varepsilon k$. Substitute $u + h$ for u and $v + k$ for v in equation (i). Subtracting equation (i) from the resulting equation we obtain

$$\Delta_{(I-\varepsilon)k}\varphi_1(u + v) = \Delta_h P(u) + \Delta_k Q(v), \quad u, v \in Y. \qquad (10.21)$$

Substituting $v = 0$ into equation (10.21) and subtracting the resulting equation from (10.21) we obtain

$$\Delta_{(I-\varepsilon)k}\Delta_v \varphi_1(u) = \Delta_k Q(v) - \Delta_k Q(0), \quad u, v \in Y. \qquad (10.22)$$

Let t be an arbitrary element of the group Y. Substitute $u + t$ for u in equation (10.22) and subtract equation (10.22) from the resulting equation. We get that the function $\varphi_1(y)$ satisfies the equation

$$\Delta_{(I-\varepsilon)k}\Delta_v \Delta_t \varphi_1(u) = 0, \quad k, v, t, u \in Y.$$

Putting here $v = t = h$ we get (ii).

Set $v' = \varepsilon v$. Then equation (i) becomes

$$\varphi_1(u + \varepsilon^{-1}v') + \varphi_2(u + v') = P(u) + Q(v'), \quad u, v' \in Y.$$

Note that $(I - \varepsilon^{-1})Y = (I - \varepsilon)Y$. Arguing as above we prove that the function $\varphi_2(y)$ also satisfies equation (ii). □

Lemma 10.10. *Let Y be an arbitrary Abelian group. Assume that a function $\varphi(y)$ satisfies equation*

$$\text{(i)} \quad \Delta_h^3 \varphi(u) = 0, \quad h, u \in Y$$

and conditions

$$\text{(ii)} \quad \varphi(-y) = \varphi(y), \quad \varphi(0) = 0, \quad y \in Y.$$

Then the function $\varphi(y)$ satisfies equation 2.16 (ii).

Proof. By Theorem 5.5 the polynomial $\varphi(y)$ can be represented in the form 5.5 (i). Since $\varphi(y)$ is a polynomial of degree ≤ 2, representation 5.5 (i) is of the form

$$\varphi(y) = g_2(y, y) + g_1(y) + g_0, \quad y \in Y,$$

where $g_2(y_1, y_2)$ is a 2-additive function, $g_1(y)$ is an additive function, and $g_0 =$ const. It follows from (ii) that $\varphi(y) = g_2(y, y)$, and hence the function $\varphi(y)$ satisfies equation 2.16 (ii). □

We use the following theorem in the sequel and it is of independent interest.

Theorem 10.11. *Let topological automorphisms α_j, β_j, $j = 1, 2$, of a group X satisfy the condition*

$$\text{(i)} \quad I - \alpha_1{\beta_1}^{-1}\beta_2{\alpha_2}^{-1} \in \mathrm{Aut}(X).$$

Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Assume that the characteristic functions $\hat{\mu}_j(y)$ do not vanish. Then if the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, then $\mu_j \in \Gamma(X)$, $j = 1, 2$.

Proof. The reasoning used at the beginning of the proof of Theorem 10.3 allows us to prove the theorem assuming that $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_j \in \mathrm{Aut}(X)$. We also note that if $\alpha \in \mathrm{Aut}(X)$, then linear forms L_1 and L_2 are independent if and only if the linear forms L_1 and αL_2 are independent. Thus in proving the theorem, we may assume without loss of generality that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$. In this case condition (i) becomes the condition $I - \delta \in \mathrm{Aut}(X)$.

Put $\varepsilon = \tilde{\delta}$. By Lemma 10.1 the independence of L_1 and L_2 yields that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + \varepsilon v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(\varepsilon v), \quad u, v \in Y. \tag{10.23}$$

Put $\varphi_j(y) = -\ln|\hat{\mu}_j(y)|$, $j = 1, 2$. We deduce from (10.23) that the functions $\varphi_j(y)$ satisfy equation 10.9 (i), where $P(u) = \varphi_1(u) + \varphi_2(u)$, $Q(v) = \varphi_1(v) + \varphi_2(\varepsilon v)$. By Lemma 10.9 the function $\varphi_1(y)$ satisfies equation 10.9 (ii). Since k is an arbitrary element of the group Y and $I-\varepsilon \in \mathrm{Aut}(Y)$, we conclude that $(I-\varepsilon)k$ is also an arbitrary element of the group Y. Hence putting $(I-\varepsilon)k = h$ in 10.9 (ii) we obtain that the function $\varphi_1(y)$ satisfies equation 10.10 (i). Obviously, the function $\varphi_1(y)$ also satisfies conditions 10.10 (ii). By Lemma 10.10 the function $\varphi_1(y)$ satisfies equation 2.16 (ii).

Arguing as above we prove that the function $\varphi_2(y)$ also satisfies equation 2.16 (ii).

The theorem will be proved if we verify that the quotient $l_j(y) = \hat{\mu}_j(y)/|\hat{\mu}_j(y)|$ is a character of the group Y. Put $\pi = (\varepsilon - I)^{-1}$, $\rho = \varepsilon\pi$, $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$. We also agree that $\alpha^0 = I$ for all $\alpha \in \mathrm{Aut}(Y)$. Substituting $u = \rho y$, $v = -\pi y$ and then $u = -\pi y$, $v = \pi y$ into (10.23) and taking into account 2.7 (d) we obtain respectively

$$f(y) = f(\rho y)f(-\pi y)|g(\rho y)|^2, \quad y \in Y, \tag{10.24}$$

and

$$g(y) = |f(\pi y)|^2 g(-\pi y)g(\rho y), \quad y \in Y. \tag{10.25}$$

We deduce from (10.24) and (10.25) that

$$f(y) = \prod_{k=0}^{n} (f((-1)^k \pi^k \rho^{n-k} y))^{C_n^k} G_n(y), \quad y \in Y, \tag{10.26}$$

where $G_n(y) > 0$, and

$$|f(y)| = \prod_{k=0}^{n} |f(\pi^k \rho^{n-k} y)|^{a(n,k)} |g(\pi^k \rho^{n-k} y)|^{b(n,k)}, \quad y \in Y. \tag{10.27}$$

Equality (10.27) implies the inequality

$$\begin{aligned} |f(z)| \le \prod_{k=0}^{n+2} |f(\pi^k \rho^{n+2-k} z)|^{a(n+2,k)} &\le \prod_{k=1}^{n+1} |f(\pi^k \rho^{n+2-k} z)|^{a(n+2,k)} \\ &= \prod_{k=0}^{n} |f(\pi^{k+1} \rho^{n+1-k} z)|^{a(n+2,k+1)}. \end{aligned} \tag{10.28}$$

It is not difficult to check that

$$a(n+2, k+1) \ge n C_n^k, \quad k = 0, 1, \ldots, n;\ n = 0, 1, 2, \ldots. \tag{10.29}$$

Set $y = \pi\rho z$. Taking into account (10.28) and (10.29), we find

$$\begin{aligned} |f(z)|^{1/n} &\le \Big(\prod_{k=0}^{n} |f(\pi^{k+1}\rho^{n+1-k}z)|^{a(n+2,k+1)}\Big)^{1/n} \\ &\le \prod_{k=0}^{n} |f(\pi^{k+1}\rho^{n+1-k}z)|^{C_n^k} \\ &= \prod_{k=0}^{n} |f(\pi^{k}\rho^{n-k}y)|^{C_n^k}. \end{aligned} \tag{10.30}$$

Inequality (10.30) yields that

$$\lim_{n\to\infty} \prod_{k=0}^{n} |f(\pi^{k}\rho^{n-k}y)|^{C_n^k} = 1. \tag{10.31}$$

It follows from (10.26) and (10.30) that

$$\lim_{n\to\infty} G_n(y) = |f(y)|. \tag{10.32}$$

Hence (10.26) and (10.32) imply

$$l_1(y) = \frac{f(y)}{|f(y)|} = \lim_{n\to\infty} \prod_{k=0}^{n} (f((-1)^k\pi^{k}\rho^{n-k}y))^{C_n^k}.$$

We conclude from this that $l_1(y)$ is a positive definite function because it is a limit of a sequence of positive definite functions. Since $|l_1(y)| = 1$ for all $y \in Y$, 2.7 (e) yields that $l_1(y) = (x_1, y)$, $x_1 \in X$. Arguing as above we prove that $l_2(y) = (x_2, y)$, $x_2 \in X$. □

Remark 10.12. If X is a group with unique division by 2, then the automorphism $\delta = -I$ satisfies the condition $I - \delta = f_2 \in \mathrm{Aut}(X)$. Hence the statement formulated in Remark 7.12 follows from Theorem 10.11.

Remark 10.13. In relation to Theorem 10.11 we will give an example of a group X which has the properties:

(a) $I - \delta \notin \mathrm{Aut}(X)$ for all $\delta \in \mathrm{Aut}(X)$;

(b) for any $\delta \in \mathrm{Aut}(X)$ if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions and the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent, then $\mu_j \in \Gamma(X)$, $j = 1, 2$.

Consider the group $X = \Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (3, 3, \dots, 3, \dots)$. Then

$$Y \cong H_{\boldsymbol{a}} = \left\{\tfrac{m}{3^n} : n = 0, 1, 2, \dots; m \in \mathbb{Z}\right\}.$$

It follows from this that the group $H_{\boldsymbol{a}}$ has the type $\mathbf{t}(H_{\boldsymbol{a}}) = (0, \infty, 0, 0, \dots)$. It is obvious that $\mathrm{Aut}(Y) = \{\pm f_3^k : k \in \mathbb{Z}\}$ (see 1.14 (d)), and hence $\mathrm{Aut}(X) = \{\pm f_3^k : k \in \mathbb{Z}\}$. Let $\delta \in \mathrm{Aut}(X)$ be an arbitrary automorphism. Since the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Theorem 10.3 if the linear forms L_1 and L_2 are independent, then $\mu_1, \mu_2 \in \Gamma(X)$. On the other hand, it is easy to see that $I - \delta \notin \mathrm{Aut}(X)$ for all $\delta \in \mathrm{Aut}(X)$.

In the rest of this section we describe groups X that have the following property: there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ such that if ξ_j are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the independence of the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ implies that all distributions $\mu_j \in \Gamma(X)$. We need the following simple lemma.

Lemma 10.14. *Let $X \neq \{0\}$ and $X \ncong \mathbb{Z}(2)$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ such that $\alpha \neq I$.*

Proof. If not all nonzero elements of the group X have order 2, then $I \neq -I$ and we put $\alpha = -I$. If every nonzero element of the group X has order 2, then by Theorem 1.11.5 the group X is topologically isomorphic to the group

$$\mathbb{Z}(2)^{\mathfrak{n}} \times \mathbb{Z}(2)^{\mathfrak{m}*},$$

where $\mathfrak{n}$ and $\mathfrak{m}$ are arbitrary cardinal numbers, $\mathbb{Z}(2)^{\mathfrak{n}}$ is considered in the product topology, and $\mathbb{Z}(2)^{\mathfrak{m}*}$ is considered in the discrete topology. It follows from this that the group X can be represented in the form $X = X_1 \times X_2 \times X_3$, where $X_1 \cong X_2 \cong \mathbb{Z}(2)$. Denote by $x = (x_1, x_2, x_3)$, $x_1, x_2 \in \mathbb{Z}(2)$, $x_3 \in X_3$, elements of the group X, and put $\alpha(x_1, x_2, x_3) = (x_2, x_1, x_3)$. □

Theorem 10.15. *Let $X \ncong \mathbb{T}$ and $X \ncong \mathbb{T} \times \mathbb{Z}(2)$. Then there exists a nonidentity automorphism $\delta \in \mathrm{Aut}(X)$ with the property: if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ yields that $\mu_j \in \Gamma(X)$, $j = 1, 2$.*

Proof. There are three possible cases:

1. The group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. In this case by Theorem 10.3 the independence of any linear forms L_1 and L_2 implies that all $\mu_j \in \Gamma(X)$.

2. The group X contains a subgroup G_1 topologically isomorphic to the circle group $\mathbb{T}$ and X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. By Theorem 1.17.1 the subgroup G_1 is a topological direct factor of the group X, i.e., $X = G_1 \times G_2$, where the subgroup G_2 contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Denote by $x = (g_1, g_2), g_j \in G_j$, $j = 1, 2$, elements of the group X.

By the condition of the theorem $G_2 \neq \{0\}$ and $G_2 \ncong \mathbb{Z}(2)$. By Lemma 10.14 there exists an automorphism $\alpha \in \mathrm{Aut}(G_2)$ such that $\alpha \neq I$. Consider an automorphism

$\delta \in \mathrm{Aut}(X)$ of the form $\delta(g_1, g_2) = (g_1, \alpha g_2)$. We will check that if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ implies that $\mu_j \in \Gamma(X)$, $j = 1, 2$.

Put $\varepsilon = \tilde{\delta}$. By Lemma 10.1 it follows from the independence of L_1 and L_2 that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.23). Since $X = G_1 \times G_2$, by Theorem 1.7.1, $Y = H_1 \times H_2$, where $H_j \cong G_j^*$, $j = 1, 2$. Taking into account that $\delta x = x$ for all $x \in G_1$, we conclude that $\varepsilon y = y$ for all $y \in H_1$. Therefore the restriction of equation (10.23) to the subgroup H_1 is of the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(v), \quad u, v \in H_1.$$

Hence the restriction of the function $f(y) = \hat{\mu}_1(y)\hat{\mu}_2(y)$ to the subgroup H_1 is a character of the subgroup H_1. It follows from this that $|\hat{\mu}_j(y)| = 1$ for all $y \in H_1$. By 2.7 (e) there exist elements $x_j \in X$ such that $\hat{\mu}_j(y) = (x_j, y)$, $y \in H_1$, $j = 1, 2$. Consider the distributions $\mu'_j = \mu_j * E_{-x_j}$. Then by 2.7 (c), $\hat{\mu}'_j(y) = 1$ for all $y \in H_1$, $j = 1, 2$. In view of Proposition 2.13 this implies that $\sigma(\mu'_j) \subset A(X, H_1) = G_2$. The characteristic functions $\hat{\mu}'_j(y)$ also satisfy equation (10.23). Note that $\varepsilon y = \tilde{\alpha} y$ for all $y \in H_2$. Consider the linear forms $L'_1 = \xi'_1 + \xi'_2$ and $L'_2 = \xi'_1 + \alpha\xi'_2$, where ξ'_j are independent random variables with values in the group G_2 and distributions μ'_j. Taking into account that the restrictions of the characteristic functions $\hat{\mu}'_j(y)$ to the subgroup H_2 also satisfy equation (10.23), by Lemma 10.1 the linear forms L'_1 and L'_2 are independent. Since $H_2 \cong G_2^*$ and the group G_2 contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, we can apply Theorem 10.3 to the group G_2. We get that $\mu'_j \in \Gamma(G_2)$. Hence $\mu_j \in \Gamma(X)$, $j = 1, 2$.

3. The group X contains a subgroup G_2 topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. By Theorem 1.17.1 the subgroup G_2 is a topological direct factor of the group X, i.e., $X = G_1 \times G_2$. Assume that an automorphism $\alpha \in \mathrm{Aut}(\mathbb{T}^2)$ is defined by the matrix $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ (see 1.13 (e)). Since $\det(I - \alpha) = 1$, we have $I - \alpha \in \mathrm{Aut}(\mathbb{T}^2)$. Taking into account that $G_2 \cong \mathbb{T}^2$, we can assume that $\alpha \in \mathrm{Aut}(G_2)$. We retain the same notation for elements of the group X as in case 2. Consider an automorphism $\delta \in \mathrm{Aut}(X)$ of the form $\delta(g_1, g_2) = (g_1, \alpha g_2)$. Then δ is the desired automorphism. Arguing as in case 2 and using Theorem 10.11 instead of Theorem 10.3 we verify that if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ implies that $\mu_j \in \Gamma(X)$, $j = 1, 2$. □

Theorem 10.15 is sharp. Namely, the following statement is true.

Proposition 10.16. *Let either $X = \mathbb{T}$ or $X = \mathbb{T} \times \mathbb{Z}(2)$. Assume that automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, $n \geq 2$, have the property that not all automorphisms $\alpha_j\beta_j^{-1}$ are equal. Then there exist independent random variables ξ_j with values in X and distributions $\mu_j \notin \Gamma(X)$ with non-vanishing characteristic functions such that the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent.*

Proof. Obviously, the group $\mathrm{Aut}(X)$ consists of two automorphisms $\mathrm{Aut}(X) = \{\pm I\}$. We restrict ourselves to proving the proposition for the circle group $X = \mathbb{T}$. Then $Y \cong \mathbb{Z}$. We will suppose without loss of generality that $Y = \mathbb{Z}$ and L_1, L_2 have the form $L_1 = \xi_1 + \cdots + \xi_n$, $L_2 = \xi_1 + \cdots + \xi_m - \xi_{m+1} - \cdots - \xi_n$, $1 \le m < n$. Consider on the group $\mathbb{Z}$ the functions

$$f(k) = \begin{cases} \exp\{-ak^2/m\} & \text{if } k \in \mathbb{Z}^{(2)}, \\ \exp\{-(ak^2 - 1)/m\} & \text{if } k \notin \mathbb{Z}^{(2)} \end{cases}$$

and

$$g(k) = \begin{cases} \exp\{-ak^2/(n-m)\} & \text{if } k \in \mathbb{Z}^{(2)}, \\ \exp\{-(ak^2 + 1)/(n-m)\} & \text{if } k \notin \mathbb{Z}^{(2)}. \end{cases}$$

Take $a > 0$ in such a way that the inequalities

$$\rho(e^{it}) = \sum_{k=-\infty}^{\infty} f(k)e^{-ikt} > 0, \quad \tau(e^{it}) = \sum_{k=-\infty}^{\infty} g(k)e^{-ikt} > 0$$

are satisfied. It follows from this that $f(k)$ and $g(k)$ are the characteristic functions of some distributions of $\mu, \nu \in \mathrm{M}^1(\mathbb{T})$ with the densities $\rho(e^{it})$ and $\tau(e^{it})$ with respect to the Haar distribution $m_{\mathbb{T}}$. Let ξ_j be independent random variables with values in the circle group $\mathbb{T}$ and distributions $\mu_j = \mu$, $j = 1, 2, \ldots, m$ and $\mu_j = \nu$, $j = m+1, m+2, \ldots, n$. It is obvious that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\prod_{j=1}^{m} \hat{\mu}_j(k+l) \prod_{j=m+1}^{n} \hat{\mu}_j(k-l) = \prod_{j=1}^{n} \hat{\mu}_j(k) \prod_{j=1}^{m} \hat{\mu}_j(l) \prod_{j=m+1}^{n} \hat{\mu}_j(-l), \quad k, l \in \mathbb{Z}.$$

By Lemma 10.1 the linear forms L_1 and L_2 are independent. It is obvious that all $\mu_j \notin \Gamma(\mathbb{T})$. □

We will also prove the following statement.

Proposition 10.17. *Assume that X is the same group as in Theorem 10.15, i.e., $X \not\cong \mathbb{T}$ and $X \not\cong \mathbb{T} \times \mathbb{Z}(2)$. Assume that the group X satisfies the conditions:*

(i) $c_X \neq \{0\}$,

(ii) $c_X \not\cong \mathbb{T}$.

Then the automorphism $\delta \in \mathrm{Aut}(X)$ in Theorem 10.15 can be chosen in such a way that the following statement holds:

(I) *There exist nondegenerate distributions μ_1 and $\mu_2 \in \Gamma(X)$ such that if ξ_j are independent random variables with values in X and distributions μ_j, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent.*

Proof. To prove the proposition we follow the scheme of the proof of Theorem 10.15. Consider the same three cases.

1. The group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Since by the condition of the proposition $c_X \neq \{0\}$, Remark 3.12 implies that there exists a nondegenerate distribution $\gamma \in \Gamma(X)$. It follows from 2.16 (ii) that the characteristic function $\hat{\gamma}(y)$ satisfies the equation

$$\hat{\gamma}(u+v)\hat{\gamma}(u-v) = \hat{\gamma}^2(u)\hat{\gamma}(v)\hat{\gamma}(-v), \quad u, v \in Y. \tag{10.33}$$

Put $\delta = -I$ and $\mu_1 = \mu_2 = \gamma$. By Lemma 10.1 equation (10.33) yields that if ξ_1 and ξ_2 are independent identically distributed random variables with values in X and distribution γ, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent.

2. The group X contains a subgroup G_1 topologically isomorphic to the circle group $\mathbb{T}$, and X contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. By Theorem 1.17.1 the subgroup G_1 is a topological direct factor of the group X, i.e., $X = G_1 \times G_2$, where the subgroup G_2 contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. It follows from (ii) that $c_{G_2} \neq \{0\}$. Denote by $x = (g_1, g_2)$, $g_j \in G_j$, $j = 1, 2$, elements of the group X. Reasoning as in case 1 we can put $\delta(g_1, g_2) = (g_1, -g_2)$, and $\mu_1 = \mu_2 = \gamma$, where γ is a nondegenerate Gaussian distribution on G_2.

3. The group X contains a subgroup G_2 topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. By Theorem 1.17.1 the subgroup G_2 is a topological direct factor of the group X, i.e., $X = G_1 \times G_2$. Put $H_2 = G_2^*$. It follows from $G_2 \cong \mathbb{T}^2$ that $H_2 \cong \mathbb{Z}^2$. Denote by (m, n), $m, n \in \mathbb{Z}$, elements of the group H_2. Let $\alpha \in \mathrm{Aut}(G_2)$ be the same automorphism as in case 3 of Theorem 10.15. Let μ_1 and μ_2 be Gaussian distributions on the group G_2 with the characteristic functions

$$\hat{\mu}_1(m,n) = \{-\sigma(t_0 m - n)^2\}, \quad \hat{\mu}_2(m,n) = \{-\sigma t_0(t_0 m - n)^2\}, \quad (m,n) \in H_2,$$

where $t_0 = \frac{\sqrt{5}-1}{2}$. It is not difficult to check directly that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u+\tilde{\alpha}v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(\tilde{\alpha}v), \quad u, v \in H_2.$$

Let $\delta \in \mathrm{Aut}(X)$ be the same automorphism as in case 3 of Theorem 10.15. It is easy to see that if the distributions μ_1 and μ_2 are considered as Gaussian distributions on X, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (10.23). By Lemma 10.1 if ξ_j are independent random variables with values in X and distributions μ_j, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. □

Remark 10.18. The proof of the group analogue of the Skitovich–Darmois theorem under the condition that the characteristic functions of the considering distributions do not vanish (Theorem 10.3) is based on the group analogue of the Cramér theorem (Theorem 4.6). As in the classical situation, the group analogue of the Cramér theorem can be obtained from the following particular case of the Skitovich–Darmois theorem.

(α) *Let* ξ_j, $j = 1, 2, 3, 4$, *be independent random variables with values in a group* X *and distributions* $\mu_1 = \mu_3$, $\mu_2 = \mu_4$ *with non-vanishing characteristic functions. Then if the linear forms* $L_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$ *and* $L_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4$ *are independent, then* $\mu_1, \mu_2 \in \Gamma(X)$.

Indeed, let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Assume that the sum $\xi_1 + \xi_2$ has a Gaussian distribution, i.e., in view of (2.1), $\gamma = \mu_1 * \mu_2 \in \Gamma(X)$. Take a random variable ξ_3 identically distributed with ξ_1 and ξ_4 identically distributed with ξ_2 in such a way that all random variables ξ_j are independent. Since the random variables $\eta_1 = \xi_1 + \xi_2$ and $\eta_2 = \xi_3 + \xi_4$ have the same distribution $\gamma \in \Gamma(X)$, by Lemma 7.1 the linear forms $L_1 = (\xi_1 + \xi_2) + (\xi_3 + \xi_4)$ and $L_2 = (\xi_1 + \xi_2) - (\xi_3 + \xi_4)$ are independent. Then it follows from (α) that $\mu_1, \mu_2 \in \Gamma(X)$.

On the other hand, statement (α) follows from the group analogue of the Cramér theorem (Theorem 4.6). To prove this we note that by Lemma 10.1 if the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$ and $L_2 = \xi_1 + \xi_2 - \xi_3 - \xi_4$ are independent, then the characteristic functions $\mu_j(y)$ satisfy equation 10.1 (i) which takes the form

$$\begin{aligned} &\hat\mu_1(u+v)\hat\mu_2(u+v)\hat\mu_1(u-v)\hat\mu_2(u-v) \\ &\quad = \hat\mu_1^2(u)\hat\mu_2^2(u)\hat\mu_1(v)\hat\mu_1(-v)\hat\mu_2(v)\hat\mu_2(-v), \quad u, v \in Y. \end{aligned} \tag{10.34}$$

Put $\gamma = \mu_1 * \mu_2 * \bar\mu_1 * \bar\mu_2$. It follows from 2.7 (c), 2.7 (d), and (10.34) that the characteristic function $\hat\gamma(y)$ satisfies the equation

$$\hat\gamma(u+v)\hat\gamma(u-v) = \hat\gamma^2(u)\hat\gamma^2(v), \quad u, v \in Y. \tag{10.35}$$

In view of 2.7 (c) and 2.7 (d), $\hat\gamma(y) > 0$. Set $\varphi(y) = -\ln\hat\gamma(y)$. It follows from equation (10.35) that the function $\varphi(y)$ satisfies equation 2.16 (ii). Hence $\gamma \in \Gamma(X)$. By Theorem 4.6 this implies that $\mu_j \in \Gamma(X)$, $j = 1, 2$. Statement (α) is proved.

We see that two classes of groups X coincide: the class of groups X for which Theorem 4.6 holds and the class of groups X for which statement (α) is true.

11 Random variables with values in the two-dimensional torus $\mathbb{T}^2$

Let X be a second countable locally compact Abelian group, α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, be topological automorphisms of X. Let ξ_j be independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions. According to Theorem 10.3 if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, then the independence of the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ yields that all $\mu_j \in \Gamma(X)$. On the other hand if a group X contains a subgroup topologically isomorphic to $\mathbb{T}$, then this statement is not true (see Remark 10.4). Therefore the following natural problem arises for such groups: to describe possible distributions μ_j of independent random variables ξ_j provided that the linear forms L_1 and L_2 are independent. In this section we solve this problem for two independent random variables taking values in the two-dimensional torus $X = \mathbb{T}^2$. There follows from the main result of this section a complete description of topological automorphisms α_j, β_j of the group $\mathbb{T}^2$ which have the property: if the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, then the random variables ξ_j are Gaussian. It turns out that these Gaussian distributions are not arbitrary. They are either degenerate or concentrated on the cosets of dense one-parameter subgroups of $\mathbb{T}^2$ (dense windings of the two-dimensional torus $\mathbb{T}^2$).

11.1 Notation. Let $X = \mathbb{T}^2$. Denote by $x = (z, w)$, $z, w \in \mathbb{T}$, elements of the group X. We have $Y \cong \mathbb{Z}^2$. To avoid introducing new notation we will assume that $Y = \mathbb{Z}^2$. Denote by $y = (m, n)$, $m, n \in \mathbb{Z}$, elements of the group Y. It follows from Definition 3.1 that the characteristic function of a Gaussian distribution on the two-dimensional torus $\mathbb{T}^2$ has the form

$$\hat{\gamma}(y) = (x, y)\exp\{-\langle Ay, y\rangle\}, \quad y = (m, n) \in \mathbb{Z}^2,$$

where $x \in \mathbb{T}^2$, $A = (a_{ij})_{i,j=1}^2$ is a symmetric positive semidefinite matrix. According to 1.14 (e) every automorphism $\delta \in \mathrm{Aut}(\mathbb{T}^2)$ is defined by an integer-valued matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $|ad - bc| = 1$. The automorphism δ acts on $\mathbb{T}^2$ as follows:

$$\delta(z, w) = (z^a w^c, z^b w^d), \quad (z, w) \in \mathbb{T}^2.$$

The adjoint automorphism $\varepsilon = \tilde{\delta} \in \mathrm{Aut}(\mathbb{Z}^2)$ is of the form

$$\varepsilon(m, n) = (am + bn, cm + dn), \quad (m, n) \in \mathbb{Z}^2.$$

We will identify the automorphisms δ and ε with the corresponding matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. Assume that α_j, β_j, $j = 1, 2$, are topological automorphisms of X. As has been noted at the beginning of the proof of Theorem 10.11, the study of possible distributions μ_j provided that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent is reduced to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$.

To prove the main theorem of this section we need some lemmas.

Lemma 11.2. *Assume that* $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Aut}(\mathbb{Z}^2)$ *and* λ_1, λ_2 *are the eigenvalues of the matrix* ε. *Consider the equation*

$$\text{(i)} \quad A + B\varepsilon = 0,$$

where $A = (a_{ij})_{i,j=1}^2$ *and* $B = (b_{ij})_{i,j=1}^2$ *are symmetric positive semidefinite matrices. Then the following statements hold:*

(Ia) *If* $\lambda_1\lambda_2 = 1$ *and* $\lambda_1 + \lambda_2 > -2$, *then equation* (i) *has a unique solution* $A = B = 0$.

(Ib) *If* $\lambda_1\lambda_2 = -1$ *and* $\lambda_1 + \lambda_2 \neq 0$, *then equation* (i) *has nonzero solutions, and any nonzero solution of equation* (i) *has the properties* $\det A = \det B = 0$ *and* $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

(IIa) *If either* $\lambda_1 = \lambda_2 = -1$ *and* $\varepsilon \neq -I$ *or* $\lambda_1 = 1$, $\lambda_2 = -1$, *then equation* (i) *has nonzero solutions, and any solution of equation* (i) *has the properties* $A = B$, $\det A = 0$, *and* $\operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$.

(IIb) *If either* $\varepsilon = -I$ *or* $\lambda_1\lambda_2 = 1$ *and* $\lambda_1 + \lambda_2 < -2$, *then there exists a solution of equation* (i) *such that* $\det A = \det B > 0$.

Proof. First note that equation (i) has a nonzero solution if and only if the following system of equations and inequalities in b_{ij} has a nonzero solution:

$$\begin{cases} b_{11} \geq 0, \qquad b_{22} \geq 0, & (11.1)\\ b_{11}b_{22} - b_{12}^2 \geq 0, & (11.2)\\ b_{11}b + b_{12}d = b_{12}a + b_{22}c, & (11.3)\\ b_{11}a + b_{12}c \leq 0, \qquad b_{12}b + b_{22}d \leq 0, & (11.4)\\ (b_{11}a + b_{12}c)(b_{12}b + b_{22}d) - (b_{12}a + b_{22}c)^2 \geq 0. & (11.5)\end{cases}$$

The solving of this system is rather cumbersome, and we divide it into several steps.

1. $\det \varepsilon = 1$, i.e.,

$$ad - bc = 1. \tag{11.6}$$

In view of (11.6) the inequalities (11.2) and (11.5) are equivalent.

A. $a \neq d$. In this case $bc \neq 0$. From (11.3) we find

$$b_{12} = (b_{11}b - b_{22}c)/(a - d). \tag{11.7}$$

Substituting (11.7) into (11.2) and taking into account (11.6) we get

$$b^2b_{11}^2 - (a^2 + d^2 - 2)b_{11}b_{22} + c^2b_{22}^2 \leq 0. \tag{11.8}$$

It follows from (11.8) that either $b_{11} = b_{22} = 0$, and then $A = B = 0$ or $b_{11}b_{22} \neq 0$. Assume that $b_{11}b_{22} \neq 0$. Put $t = b_{11}/b_{22}$. We obtain from (11.8)

$$b^2t^2 - (a^2 + d^2 - 2)t + c^2 \leq 0. \tag{11.9}$$

Consider the equation

$$b^2t^2 - (a^2 + d^2 - 2)t + c^2 = 0, \tag{11.10}$$

and denote by $D = (a - d)^2((a + d)^2 - 4)$ its discriminant.

(i) $(a + d)^2 < 4$. Then $D < 0$.

Equation (11.10) and hence inequality (11.9) has no solutions. So, equation (i) has a unique solution $A = B = 0$.

(ii) $a + d = 2$. This implies that $D = 0$ and $a \neq 1$. Equation (11.10) and hence inequality (11.9) has a unique solution $t = (a - 1)^2/b^2$. Substituting the expression for b_{12} from (11.7) into the second inequality of system (11.4) and taking into account (11.6) we get

$$(b_{11}b^2 - b_{22}(a^2 - 4a + 3))/2(a - 1) \le 0,$$

but this contradicts the equality $b_{11}/b_{22} = (a-1)^2/b^2$. Hence equation (i) has a unique solution $A = B = 0$.

(iii) $a + d = -2$. This implies that $D = 0$ and $a \neq -1$. Equation (11.10) and hence inequality (11.9) has a unique solution $t = (a+1)^2/b^2$. Substituting the expression for b_{12} from (11.7) into system (11.4) and taking into account (11.6) we get

$$\frac{b_{11}(a^2-1) - b_{22}c^2}{2(a+1)} \le 0, \quad \frac{b_{11}b^2 - b_{22}(a^2 + 4a + 3)}{2(a+1)} \le 0. \tag{11.11}$$

Since

$$b_{11}/b_{22} = (a+1)^2/b^2, \tag{11.12}$$

it is easy to see that system of inequalities (11.11) holds for arbitrary $b_{11} > 0$, $b_{22} > 0$ such that (11.12) is satisfied. In this case all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} (a+1)^2/b^2 & (a+1)/b \\ (a+1)/b & 1 \end{pmatrix},$$

where $\sigma \ge 0$. Obviously, $\det A = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$.

(iv) $(a+d)^2 > 4$ and $a > d$. This implies that $D > 0$ and equation (11.10) has the solutions

$$\begin{aligned} t_1 &= \frac{a^2 + d^2 - 2 - (a-d)\sqrt{(a+d)^2 - 4}}{2b^2}, \\ t_2 &= \frac{a^2 + d^2 - 2 + (a-d)\sqrt{(a+d)^2 - 4}}{2b^2}. \end{aligned} \tag{11.13}$$

Hence the solutions of inequality (11.9) are $t \in [t_1, t_2]$. It is obvious that, $t_1 > 0$. Substituting the expression for b_{12} from (11.7) into system (11.4), taking into account (11.6) and the equality $t = b_{11}/b_{22}$, we deduce

$$t(a^2 - 1) - c^2 \le 0, \quad tb^2 - (d^2 - 1) \le 0. \tag{11.14}$$

If $a \in \{0, \pm 1\}$, then the first inequality of system (11.14) holds true for $t \in [t_1, t_2]$, and system (11.14) on the segment $t \in [t_1, t_2]$ is equivalent to the inequality

$$t \le (d^2 - 1)/b^2. \tag{11.15}$$

Suppose that $a \notin \{0, \pm 1\}$. Observe that (11.6) implies the inequality $(d^2-1)/b^2 \le c^2/(a^2-1)$, and hence in this case system (11.14) is also equivalent to inequality (11.15).

Obviously, if $a + d > 2$, then $t_1 > (d^2-1)/b^2$. Hence inequality (11.15) has no solutions on the interval $[t_1, t_2]$, so that equation (i) has a unique solution $A = B = 0$.

If $a+d < -2$, then $t_2 < (d^2-1)/b^2$. Therefore inequality (11.14) is fulfilled for any $t \in [t_1, t_2]$. It follows from what has been said that all nonzero solutions of equation (i) are of the form

$$B = \sigma \begin{pmatrix} t & (bt-c)/(a-d) \\ (bt-c)/(a-d) & 1 \end{pmatrix}, \quad A = -B\varepsilon,$$

where $t_1 \le t \le t_2, \sigma > 0$. Furthermore, if $t_1 < t < t_2$, then $\det A = \det B > 0$, and if either $t = t_1$ or $t = t_2$, then $\det A = \det B = 0$. We conclude from (11.13) that t_j are irrational numbers, and hence in the latter case $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

(v) $(a+d)^2 > 4$ and $a < d$. Set $t = b_{22}/b_{11}$ and pass from inequality (11.8) to the inequality $c^2t^2 - (a^2+d^2-2)t + b^2 \le 0$. Arguing as in case (iv), we obtain that if $a+d > 2$, then equation (i) has a unique solution $A = B = 0$, and if $a+d < -2$, then equation (i) has a solution such that $\det A = \det B > 0$.

B. $a = d$. We conclude from (11.3) that

$$b_{11}b = b_{22}c. \tag{11.16}$$

(i) $bc < 0$. This implies that $a = 0$. From (11.16) we find $b_{11} = b_{22} = 0$. Hence equation (i) has a unique solution $A = B = 0$.

(ii) $bc > 0$. This implies that $|a| \ge 2$. Assume that $a \ge 2$. It follows from the first inequality of system (11.4) that $b_{11} \le -cb_{12}/a$. Hence, in view of (11.16), (11.2), and (11.6) we get $b_{11}^2 \le c^2b_{12}^2/a^2 \le (a^2-1)b_{11}^2/a^2$. So, $b_{11} = b_{22} = 0$ and hence equation (i) has a unique solution $A = B = 0$.

Suppose that $a \le -2$. We note that if $b_{11} = 0$, then $b_{22} = 0$ and hence $A = B = 0$. Let $b_{11} \ne 0$. Set $s = b_{12}/b_{11}$. Taking into consideration (11.16) we can rewrite the system of inequalities (11.2) and (11.4) in the form

$$s^2 \le b/c, \quad a + cs \le 0. \tag{11.17}$$

We conclude from (11.6) that the solutions of system (11.17) are $s \in [-\sqrt{b/c}, \sqrt{b/c}]$. It follows from what has been said that all nonzero solutions of equation (i) are of the form

$$B = \sigma \begin{pmatrix} 1 & s \\ s & b/c \end{pmatrix}, \quad A = -B\varepsilon,$$

where $-\sqrt{b/c} \le s \le \sqrt{b/c}, \sigma > 0$. Furthermore, if $-\sqrt{b/c} < s < \sqrt{b/c}$, then $\det A = \det B > 0$, and if $s = \pm\sqrt{b/c}$, then $\det A = \det B = 0$. In the latter case the irrationality of $\sqrt{b/c}$ implies that $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

(iii) $b = 0, c \ne 0$. We note that in this case $a = \pm 1$. We deduce from (11.16) that $b_{22} = 0$ and hence $b_{12} = 0$. Therefore it follows from system (11.4) that $b_{11}a \le 0$. Taking this into account, we get that $b_{11} = 0$ for $a = 1$. So, equation (i) has a unique solution $A = B = 0$. If $a = -1$, then all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{11.18}$$

where $\sigma \geq 0$. Obviously, $\det A = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$.

(iv) $c = 0, b \neq 0$. We note that in this case $a = \pm 1$. We conclude from (11.16) that $b_{11} = 0$ and hence $b_{12} = 0$. Therefore it follows from system (11.4) that $b_{22}a \leq 0$. Taking this into account, we get that $b_{22} = 0$ for $a = 1$. So, equation (i) has a unique solution $A = B = 0$. If $a = -1$, then all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{11.19}$$

where $\sigma \geq 0$. Obviously, $\det A = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$.

(v) $b = c = 0$. In this case $\varepsilon = \pm I$, and the statement of the lemma is obvious.

2. $\det \varepsilon = -1$, i.e.,

$$ad - bc = -1. \tag{11.20}$$

We deduce from (11.20) that if the matrices A and B satisfy equation (i), then $\det A = \det B = 0$. Hence inequality (11.2) becomes the equation

$$b_{11}b_{22} - b_{12}^2 = 0, \tag{11.21}$$

and inequality (11.5) becomes the equation

$$(b_{11}a + b_{12}c)(b_{12}b + b_{22}d) - (b_{12}a + b_{22}c)^2 = 0. \tag{11.22}$$

Equations (11.21) and (11.22) are equivalent. Equation (i) has a nonzero solution if and only if the system of equations and inequalities (11.1), (11.3), (11.4), and (11.21) has a nonzero solution.

A. $a \neq \pm d$. Note that $a \neq -d$ implies $bc \neq 0$. Substituting (11.7) into (11.21) and taking into account (11.20), we obtain

$$b^2 b_{11}^2 - (a^2 + d^2 + 2)b_{11}b_{22} + c^2 b_{22}^2 = 0. \tag{11.23}$$

We conclude from (11.23) that either $b_{11} = b_{22} = 0$, and then $A = B = 0$ or $b_{11}b_{22} \neq 0$. Assume that $b_{11}b_{22} \neq 0$.

(i) $a < d$. Put $t = b_{11}/b_{22}$. We get from (11.23)

$$b^2 t^2 - (a^2 + d^2 + 2)t + c^2 = 0. \tag{11.24}$$

The solutions of equation (11.24) are of the form

$$\begin{aligned} t_1 &= \frac{a^2 + d^2 + 2 - (a - d)\sqrt{(a + d)^2 + 4}}{2b^2}, \\ t_2 &= \frac{a^2 + d^2 + 2 + (a - d)\sqrt{(a + d)^2 + 4}}{2b^2}. \end{aligned} \tag{11.25}$$

In view of (11.20), (11.7), and the equality $t = b_{11}/b_{22}$ we can rewrite the system (11.4) in the form

$$t \geq c^2/(a^2 + 1), \quad t \geq (d^2 + 1)/b^2. \tag{11.26}$$

We deduce from (11.20) that $c^2/(a^2+1) \le (d^2+1)/b^2$. Hence system (11.26) is equivalent to the inequality $t \ge (d^2+1)/b^2$. As is easily seen, inequality (11.26) is true for $t = t_1$ and it is false for $t = t_2$. It follows from this that equation (i) has nonzero solutions, and they are of the form

$$B = \sigma \begin{pmatrix} t_1 & (bt_1 - c)/(a-d) \\ (bt_1 - c)/(a-d) & 1 \end{pmatrix}, \quad A = kB, \qquad (11.27)$$

where $\sigma > 0$, $k = (\sqrt{(a+d)^2+4} - a - d)/2$. It is obvious that $\det A = \det B = 0$. It follows from (11.25) that t_1 is an irrational number. Hence $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

(ii) $a > d$. Put $t = b_{22}/b_{11}$ and pass from equation (11.23) to the equation

$$c^2t^2 - (a^2+d^2+2)t + b^2 = 0.$$

Arguing as in case (i) we prove that equation (i) has nonzero solutions, each of which has the properties $\det A = \det B = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

B. $a = d \ne 0$. We note that in this case $bc > 0$. We conclude from (11.16) and (11.21) that $b_{12}^2 - bb_{11}^2/c = 0$ and hence $b_{12} = \pm b_{11}\sqrt{b/c}$. Taking into account system (11.4), we find $b_{12} = -b_{11}\sqrt{b/c}$. It is easily verified that equation (i) has nonzero solutions and they are of the form

$$B = \sigma \begin{pmatrix} 1 & -\sqrt{b/c} \\ -\sqrt{b/c} & b/c \end{pmatrix}, \quad A = kB,$$

where $\sigma > 0$, $k = \sqrt{a^2+1} - a$. It is obvious that $\det A = \det B = 0$. The irrationality of $\sqrt{b/c}$ implies that $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$.

C. $a = -d \ne 0$. This implies that $a \ne d$. Hence (11.7) holds and we obtain (11.23).

(i) $a < 0$, $b \ne 0$. We conclude from (11.23) that if $b_{22} = 0$, then $b_{11} = 0$ and hence $A = B = 0$. Suppose that $b_{22} \ne 0$. Putting $t = b_{11}/b_{22}$ and arguing as in case A (i) we verify that all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} (a-1)^2/b^2 & (a-1)/b \\ (a-1)/b & 1 \end{pmatrix},$$

where $\sigma \ge 0$.

(ii) $a < 0$, $b = 0$. It follows from (11.20) that $a = -1$. The second inequality of system (11.4) implies that $b_{22} = 0$ and hence $b_{12} = 0$. We see that all solutions of equation (i) are of the form (11.18).

(iii) $a > 0$, $c \ne 0$. We deduce from (11.23) that if $b_{11} = 0$, then $b_{22} = 0$ and hence $A = B = 0$. Suppose that $b_{11} \ne 0$. Setting $t = b_{22}/b_{11}$ and arguing as in case A (ii), we verify that all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} 1 & -(a+1)/c \\ -(a+1)/c & (a+1)^2/c^2 \end{pmatrix},$$

where $\sigma \ge 0$.

(iv) $a > 0$, $c = 0$. We conclude from (11.20) that $a = 1$. The first inequality of system (11.4) implies that $b_{11} = 0$ and hence $b_{12} = 0$. We see that all solutions of equation (i) are of the form (11.19).

D. $a = d = 0$. Then $bc = 1$ and hence either $b = c = 1$ or $b = c = -1$. It follows from (11.16) that $b_{11} = b_{22}$, and (11.21) implies that $b_{22}^2 = b_{12}^2$. We deduce from system (11.4) that in the case when $b = c = 1$ all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

where $\sigma \geq 0$. If $b = c = -1$, then all solutions of equation (i) are of the form

$$A = B = \sigma \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where $\sigma \geq 0$.

It is obvious that in cases **C** and **D** we have $\det A = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$.

To complete the proof of Lemma 11.2, we note that $\lambda_1 \lambda_2 = \det \varepsilon$ and $\lambda_1 + \lambda_2 = a + d$. □

Lemma 11.3. *Assume that* $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{Aut}(\mathbb{Z}^2)$, λ_1 *and* λ_2 *are the eigenvalues of the matrix* ε. *The kernel* $\operatorname{Ker}(\varepsilon - I) \neq \{0\}$ *if and only if either* $\lambda_1 = \lambda_2 = 1$ *and then* $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & 2-a \end{smallmatrix}\right)$, $\det \varepsilon = 1$, *or* $\lambda_1 = 1$, $\lambda_2 = -1$ *and then* $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, $\det \varepsilon = -1$.

Proof. Straightforward. □

Lemma 11.4. *Assume that symmetric positive semidefinite matrices* $A = (a_{ij})_{i,j=1}^2$ *and* $B = (b_{ij})_{i,j=1}^2$ *satisfy equation* 11.2 (i) *and* $\det A = 0$. *Then* $B = kA$, *where* $k > 0$.

Proof. We conclude from 11.2 (i) that the inequalities $a_{11}a_{22} \neq 0$ and $b_{11}b_{22} \neq 0$ are equivalent. Suppose that $a_{11}a_{22} \neq 0$. In this case the matrices A and B can be written in the form $A = x^2 \left(\begin{smallmatrix} t^2 & t \\ t & 1 \end{smallmatrix}\right)$ and $B = y^2 \left(\begin{smallmatrix} s^2 & s \\ s & 1 \end{smallmatrix}\right)$, where $xy \neq 0$. Suppose $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{Aut}(\mathbb{Z}^2)$. Substituting the expressions for A, B, and ε into 11.2 (i) we get

$$x^2 \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix} + y^2 \begin{pmatrix} as^2 + cs & bs^2 + ds \\ as + c & bs + d \end{pmatrix} = 0.$$

It follows from this that the equalities

$$x^2 = -y^2(bs + d), \quad x^2 t = -y^2 s(bs + d)$$

hold. Hence $t = s$ and $B = kA$, where $k > 0$. In the case when $a_{11}a_{22} = 0$ the statement of the lemma is obvious. □

We will prove now the main theorem of this section.

Theorem 11.5. *Assume that $\delta \in \mathrm{Aut}(\mathbb{T}^2)$ and λ_1, λ_2 are the eigenvalues of the matrix $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let ξ_1 and ξ_2 be independent random variables with values in the group $\mathbb{T}^2$ and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Suppose that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Then the following statements hold:*

- (Ia) *If $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 > -2$, then μ_j are degenerate distributions.*
- (Ib) *If $\lambda_1\lambda_2 = -1$ and $\lambda_1 + \lambda_2 \neq 0$, then μ_j are Gaussian distributions concentrated on shifts of a one-parameter subgroup. This subgroup is dense in $\mathbb{T}^2$.*
- (IIa) *If either $\lambda_1 = \lambda_2 = -1$ and $\delta \neq -I$ or $\lambda_1 = 1$, $\lambda_2 = -1$, then $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in \mathbb{T}^2$, $\gamma \in \Gamma(K)$, K is a subgroup of $\mathbb{T}^2$ topologically isomorphic to $\mathbb{T}$, π_j are signed measures on F, F is a subgroup of K generated by the element of order 2, and $\pi_1 * \pi_2 = E_{(1,1)}$.*
- (IIb) *If either $\delta = -I$ or $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 < -2$, then there exist independent random variables ξ_1 and ξ_2 with values in the group $\mathbb{T}^2$ and distributions $\mu_1, \mu_2 \notin \Gamma(\mathbb{T}^2)$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. In so doing, distributions μ_j have non-vanishing characteristic functions and non-vanishing densities with respect to the Haar distribution $m_{\mathbb{T}^2}$.*

Proof. Put $\varepsilon = \tilde{\delta}$, $L = \mathrm{Ker}(\varepsilon - I)$. By Lemma 10.1 the independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.23). First we will prove statements (Ia), (Ib) and (IIa). Consider two possibilities: $L \neq \{0\}$ and $L = \{0\}$.

1. $L \neq \{0\}$. Assume that $\delta \neq I$ and prove that if $L \neq \{0\}$, then there exists a subgroup K of the group $\mathbb{T}^2$, K is topologically isomorphic to the circle group $\mathbb{T}$ such that after appropriate shifts all μ_j are supported in K and the restriction of δ to K is an automorphism of the subgroup K.

Since the subgroup L is invariant with respect to ε, we can consider the restriction of equation (10.23) to L. We get

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u+v) = \hat{\mu}_1(u)\hat{\mu}_1(v)\hat{\mu}_2(u)\hat{\mu}_2(v), \quad u, v \in L. \tag{11.28}$$

Put $h(y) = \hat{\mu}_1(y)\hat{\mu}_2(y)$. We conclude from (11.28) that the function $h(y)$ satisfies the equation $h(u+v) = h(u)h(v)$ on the subgroup L, i.e., $h(y)$ is a character of the subgroup L. It follows from 2.7 (e) that the restrictions of the functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ to L are also characters of the subgroup L. We deduce from Theorem 1.9.2 that there exist elements $x_j \in \mathbb{T}^2$ such that $\hat{\mu}_j(y) = (x_j, y)$, $y \in L$, $j = 1, 2$. Replacing the distributions μ_j with their shifts we can assume without loss of generality that $\hat{\mu}_1(y) = \hat{\mu}_2(y) = 1$, $y \in L$. Then by Proposition 2.13, $\sigma(\mu_j) \subset A(\mathbb{T}^2, L)$, $j = 1, 2$. Put $K = A(\mathbb{T}^2, L)$. Taking into account that $\varepsilon L = L$, we get that ε induces some automorphism $\hat{\varepsilon}$ on the factor group $\mathbb{Z}^2/L$ by the formula $\hat{\varepsilon}[y] = [\varepsilon y]$, $y \in [y]$. It follows from Theorems 1.9.1 and 1.9.2 that $K \cong (\mathbb{Z}^2/L)^*$. As appears from above, the restriction of the automorphism δ to the subgroup K is an automorphism of the subgroup K and this restriction coincides with the automorphism adjoint to $\hat{\varepsilon}$. It remains to prove that $K \cong \mathbb{T}$.

It follows from $\varepsilon \neq I$ that $L \cong \mathbb{Z}$. Let $y_0 = (p_0, q_0)$ be a generator of the group L, i.e., $L = \{ky_0 : k \in \mathbb{Z}\}$. Obviously, p_0 and q_0 are relatively prime. This implies that there exist some integers m_0 and n_0 such that

$$p_0 n_0 - q_0 m_0 = 1. \tag{11.29}$$

Set $L' = \{kz_0 : k \in \mathbb{Z}\}$, where $z_0 = (m_0, n_0)$. Taking into account equality (11.29), we easily verify that $\mathbb{Z}^2 = L \times L'$. Since $L' \cong \mathbb{Z}$, we have $K \cong (\mathbb{Z}^2/L)^* \cong (L')^* \cong \mathbb{T}$. It means that we have reduced our problem to the case when independent random variables take values in the circle group $\mathbb{T}$.

Assume that $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 > -2$. We conclude from Lemma 11.3 and the condition $L \neq \{0\}$ that $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & 2-a \end{smallmatrix}\right)$, $\det \varepsilon = 1$. We will prove that $\hat{\varepsilon} = I$. Obviously, it suffices to show that $\hat{\varepsilon}[z_0] = [z_0]$, i.e., $\varepsilon z_0 - z_0 \in L$. To put it in another way, $\varepsilon(\varepsilon z_0 - z_0) = \varepsilon z_0 - z_0$. But this equality follows directly from $\det \varepsilon = 2a - a^2 - bc = 1$. Hence the restriction of δ to the subgroup K is the identity automorphism. We have independent random variables ξ_1 and ξ_2 with values in the subgroup K and distributions μ_1 and μ_2; moreover the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \xi_2$ are independent. It is obvious that in this case μ_j are degenerate distributions.

If $\delta = I$, then equation (10.23) becomes equation (11.28) which holds true on $\mathbb{Z}^2$. It follows from this that μ_j are degenerate distributions. Statement (Ia) in case 1 is proved.

Assume now that either $\lambda_1 = \lambda_2 = -1$ and $\varepsilon \neq -I$ or $\lambda_1 = 1, \lambda_2 = -1$. It follows from Lemma 11.3 and the condition $L \neq \{0\}$ that $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, $\det \varepsilon = -1$. We will prove that $\hat{\varepsilon} = -I$. Obviously, it suffices to show that $\hat{\varepsilon}[z_0] = -[z_0]$, i.e., $\varepsilon z_0 + z_0 \in L$. To put it in another way, $\varepsilon(\varepsilon z_0 + z_0) = \varepsilon z_0 + z_0$. But this equality follows directly from $\det \varepsilon = -a^2 - bc = -1$. Hence the restriction of the automorphism δ to the subgroup K coincides with $-I$. It follows from Corollary 8.6 that $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in \mathbb{T}^2$, $\gamma \in \Gamma(K)$, π_j are signed measures on F, where F is the subgroup of K generated by the element of order 2, and $\pi_1 * \pi_2 = E_{(1,1)}$. Statement (IIa) in case 1 is proved.

2. $L = \{0\}$. Set $H = (I - \varepsilon)\mathbb{Z}^2$. Then $H \cong \mathbb{Z}^2$. Put $\nu_j = \mu_j * \bar{\mu}_j$. We deduce from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0,\ y \in \mathbb{Z}^2$. It is obvious that the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.23). We will first prove the following statement.

(α) *If $L = \{0\}$, then on the subgroup H the characteristic functions $\hat{\nu}_j(y)$ can be represented in the form*

$$\hat{\nu}_1(y) = \exp\{-\langle Ay, y\rangle\}, \quad \hat{\nu}_2(y) = \exp\{-\langle By, y\rangle\}, \quad y \in H,$$

where A and B are symmetric positive semidefinite matrices satisfying equation 11.2 (i).

To prove this, put $\varphi_j(y) = -\ln \hat{\nu}_j(y)$. We conclude from (10.23) that the functions $\varphi_j(y)$ satisfy equation 10.9 (i). Lemma 10.9 implies that each of the functions $\varphi_j(y)$ satisfies equation 10.9 (ii). Substituting $(I - \varepsilon)k = h$ in 10.9 (ii), where h is an arbitrary element of the subgroup H, we get that on H the function $\varphi_j(y)$ satisfies equation

$$\Delta_h^3 \varphi_j(u) = 0, \quad h, u \in H.$$

Hence $\varphi_j(y) = \varphi_j(m,n)$ is a polynomial on H of degree ≤ 2. Note that $H \cong \mathbb{Z}^2$ and $\varphi_j(y) \geq 0$, $\varphi_j(-y) = \varphi_j(y)$, $y \in \mathbb{Z}^2$, $\varphi_j(0) = 0$. Taking into account Remark 5.12, it follows from this that

$$\varphi_1(m,n) = a_{11}m^2 + 2a_{12}mn + a_{22}n^2 = \langle Ay, y\rangle, \quad y = (m,n) \in H,$$
$$\varphi_2(m,n) = b_{11}m^2 + 2b_{12}mn + b_{22}n^2 = \langle By, y\rangle, \quad y = (m,n) \in H,$$

where $A = (a_{ij})_{i,j=1}^2$, $B = (b_{ij})_{i,j=1}^2$ are symmetric positive semidefinite matrices. Considering equation 10.9 (i) on H and substituting the expressions for $\varphi_j(y)$ into 10.9 (i), we obtain

$$\langle u, (A + B\varepsilon)v\rangle = 0, \quad u, v \in H. \tag{11.30}$$

Since $H \cong \mathbb{Z}^2$, (11.30) implies that the matrices A and B satisfy equation 10.3 (i). Statement (α) is proved.

Assume that $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 > -2$. Apply (α). Since the matrices A and B satisfy equation 11.2 (i), by Lemma 11.2, $A = B = 0$ and hence $\hat{\nu}_j(y) = 1$, $y \in H$. Then Proposition 2.13 implies $\sigma(\nu_j) \subset A(\mathbb{T}^2, H)$, $j = 1, 2$. Put $K = A(\mathbb{T}^2, H)$. It follows from $\varepsilon H = H$ that ε induces an automorphism $\hat{\varepsilon}$ on the factor group $\mathbb{Z}^2/H$ by the formula $\hat{\varepsilon}[y] = [\varepsilon y]$, $y \in [y]$. By Theorems 1.9.1 and 1.9.2, $K \cong (\mathbb{Z}^2/H)^*$. It follows from what has been said that the restriction of the automorphism δ to the subgroup K is an automorphism of the subgroup K and this restriction coincides with the automorphism adjoint to $\hat{\varepsilon}$. Since $H \cong \mathbb{Z}^2$, it follows from this that $\mathbb{Z}^2/H$ is a finite group. Therefore K is also a finite group. Hence the group K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let ξ_1' and ξ_2' be independent random variables with values in the group K and distributions ν_j. Applying Theorem 10.3 to the random variables ξ_j' and to the group K we find that $\nu_j \in \Gamma(K)$. By Proposition 3.6, $\Gamma(K) = D(K)$. Hence ν_j are degenerate distributions, and μ_j are also degenerate distributions. Statement (Ia) is proved in case 2, and hence it is completely proved.

To prove (Ib) and (IIa) we need the following statement.

(β) *Put $\gamma = \nu_1 * \nu_2$. Then $\gamma \in \Gamma(\mathbb{T}^2)$.*

As has been shown in the proof of Theorem 10.3, it follows from equation 10.11 (i) that the function $P(y) = \varphi_1(y) + \varphi_2(y)$ satisfies equation

$$\Delta_h^3 P(u) = 0, \quad u, h \in \mathbb{Z}^2.$$

Hence $P(y) = P(m,n)$ is a polynomial on $\mathbb{Z}^2$ of degree ≤ 2. Since $P(y) \geq 0$, $P(-y) = P(y)$, $y \in \mathbb{Z}^2$, and $P(0) = 0$, it follows from Remark 5.12 that

$$P(m,n) = c_{11}m^2 + 2c_{12}mn + c_{22}n^2 = \langle Cy, y\rangle, \quad y = (m,n) \in \mathbb{Z}^2,$$

where $C = (C_{ij})_{i,j=1}^2$ is a symmetric positive semidefinite matrix. Hence statement (β) follows from 2.7 (b) and 2.7 (c).

Assume that $\lambda_1\lambda_2 = -1$ and $\lambda_1 + \lambda_2 \neq 0$. Apply ($\alpha$). Since matrices A and B satisfy equation 11.2 (i), by Lemma 11.2 equation 11.2 (i) has nonzero solutions and any nonzero solution of equation 11.2 (i) has the properties $\det A = \det B = 0$ and $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$. It follows from this that $a_{11}a_{22} \neq 0$. We have $\langle Ay, y\rangle = (\sqrt{a_{11}}m \pm \sqrt{a_{22}}n)^2$, $y = (m,n)$. Assume for definiteness that $\langle Ay, y\rangle = (\sqrt{a_{11}}m + \sqrt{a_{22}}n)^2$. Applying ($\boldsymbol{\alpha}$), ($\beta$) and Lemma 11.4 we get $\langle Cy, y\rangle = q(\sqrt{a_{11}/a_{22}}m + n)^2$, $y \in H$, where $q > 0$. Taking into account that $H \cong \mathbb{Z}^2$, this representation also holds true for $y \in \mathbb{Z}^2$. It follows from $\operatorname{Ker} A \cap \mathbb{Z}^2 = \{0\}$ that $\sqrt{a_{11}/a_{22}}$ is an irrational number. Put $\gamma = \nu_1 * \nu_2$. Then $\hat{\gamma}(m,n) = \exp\{-q(\sqrt{a_{11}/a_{22}}m + n)^2\}$.

Consider a Gaussian distribution N on the group $\mathbb{R}$ with the characteristic function $\widehat{N}(s) = \exp\{-qs^2\}$. Let $\tau\colon \mathbb{Z}^2 \mapsto \mathbb{R}$ be the homomorphism given by $\tau(m,n) = \sqrt{a_{11}/a_{22}}m + n$. Denote by $p = \tilde{\tau}\colon \mathbb{R} \mapsto \mathbb{T}^2$ the adjoint homomorphism. It follows from Proposition 2.10 and 2.7 (b) that $\gamma = p(N)$. Since $\sqrt{a_{11}/a_{22}}$ is an irrational number, τ is a monomorphism. We conclude from 1.13 (a) and 1.13 (b) that the image $p(\mathbb{R})$ is dense in $\mathbb{T}^2$. Thus the distribution γ is concentrated on a one-parameter subgroup $V = p(\mathbb{R}) \subset \mathbb{T}^2$ such that V is dense in $\mathbb{T}^2$. Since $\gamma = \mu_1 * \bar{\mu}_1 * \mu_2 * \bar{\mu}_2$, Proposition 2.2 implies that the distributions μ_j can be replaced by their shifts μ'_j such that

$$\gamma = \mu'_1 * \bar{\mu}'_1 * \mu'_2 * \bar{\mu}'_2,$$

and μ'_j are concentrated on V. Taking into account that the image $\tau(\mathbb{Z}^2)$ is dense in $\mathbb{R}$, we deduce from 1.13 (b) that p is a monomorphism. By Corollary 2.5, $N = N_1 * \bar{N}_1 * N_2 * \bar{N}_2$, where $N_j = p^{-1}(\mu'_j)$. Applying Theorem 2.18 we get $N_j \in \Gamma(\mathbb{R})$. This implies that $\mu'_j = p(N_j) \in \Gamma(\mathbb{T}^2)$ and hence $\mu_j \in \Gamma(\mathbb{T}^2)$. Statement (Ib) is proved.

Assume now that either $\lambda_1 = \lambda_2 = -1$ and $\delta \neq -I$ or $\lambda_1 = 1$, $\lambda_2 = -1$. Apply (α). Since the matrices A and B satisfy equation 11.2 (i), by Lemma 11.2 equation 11.2 (i) has nonzero solutions, and any solution of equation 11.2 (i) has the properties $A = B$, $\det A = 0$, and $M = \operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$. It follows from $A = B$, (α), and (β) that

$$\hat{\gamma}(y) = \exp\{-2\langle Ay, y\rangle\}, \quad y \in H. \tag{11.31}$$

Taking into account that $H \cong \mathbb{Z}^2$, representation (11.31) also holds true for $y \in \mathbb{Z}^2$. Therefore $\hat{\gamma}(y) = 1$ for $y \in M$ and hence, by Proposition 2.13, $\sigma(\gamma) \subset A(\mathbb{T}^2, M)$. Put $K = A(\mathbb{T}^2, M)$. By Proposition 2.2, distributions μ_j can be replaced by their shifts μ'_j such that $\sigma(\mu'_j) \subset K$. It is obvious that $K \cong \mathbb{T}$. As is easily seen, $M \cong \mathbb{Z}$ and there exists a subgroup $M' \subset \mathbb{Z}^2$ such that $\mathbb{Z}^2 = M \times M'$. Since $A = B$, we have $A + A\varepsilon = 0$. This implies that $\varepsilon(M) \subset M$ and $\varepsilon^{-1}(M) \subset M$. Hence $\varepsilon(M) = M$. Therefore the restriction of the automorphism ε to the subgroup M is an automorphism of the subgroup M and the automorphism ε induces some automorphism $\hat{\varepsilon}$ on the factor group $\mathbb{Z}^2/M$ by the formula $\hat{\varepsilon}[y] = [\varepsilon y]$, $y \in [y]$. It follows from $A(I + \varepsilon) = 0$ that $y + \varepsilon y \in M$ for $y \in \mathbb{Z}^2$. Therefore $[y + \varepsilon y] = 0$ for $y \in \mathbb{Z}^2$, and hence $\hat{\varepsilon}[y] = -[y]$. Taking into account that $\hat{\varepsilon}$ is the adjoint automorphism of the restriction of the automorphism δ to K, we see that the restriction of the automorphism δ to K

coincides with $-I$. It follows from Corollary 8.6 that $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in \mathbb{T}^2$, $\gamma \in \Gamma(K)$, π_j are signed measures on F, where F is the subgroup of K generated by the element of order 2 and $\pi_1 * \pi_2 = E_{(1,1)}$. Statement (IIa) is proved in case 2 and hence it is completely proved.

We now prove statement (IIb). Assume that either $\delta = -I$ or $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 < -2$. By Lemma 11.2 there exist solutions of equation 11.2 (i) such that $\det A = \det B > 0$. Observe that $I - \varepsilon \notin \mathrm{Aut}(\mathbb{Z}^2)$. Let ν_j be a Gaussian distribution on the group $\mathbb{T}^2$ with characteristic function $\hat{\nu}_j(y) = \exp\{-\varphi_j(y)\}$, $j = 1, 2$, where $\varphi_1(y) = \langle Ay, y\rangle$, $\varphi_2(y) = \langle By, y\rangle$. It is obvious that the characteristic functions $\hat{\nu}_j(y)$ satisfy equation (10.23). Set $G = A(\mathbb{T}^2, H)$. It follows from $I - \varepsilon \notin \mathrm{Aut}(\mathbb{Z}^2)$ that $H \neq \mathbb{Z}^2$ and hence $G \neq \{0\}$. Take $0 < r < 1$ and consider the distribution $\pi_1 = rE_{(1,1)} + (1-r)m_G$ and the signed measure $\pi_2 = (1/r)E_{(1,1)} + (1-1/r)m_G$. By Theorem 1.9.1, $H = A(\mathbb{Z}^2, G)$, and hence the characteristic function $\hat{m}_G(y)$ is of the form

$$\hat{m}_G(y) = \begin{cases} 1 & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$$

It follows from this that the characteristic functions $\hat{\pi}_j(y)$ are of the form

$$\hat{\pi}_1(y) = \begin{cases} 1 & \text{if } y \in H, \\ r & \text{if } y \notin H, \end{cases} \quad \hat{\pi}_2(y) = \begin{cases} 1 & \text{if } y \in H, \\ 1/r & \text{if } y \notin H. \end{cases}$$

We will verify that the functions $\hat{\pi}_j(y)$ satisfy equation (10.23). It follows from $\varepsilon(H) = H$ that $\hat{\pi}_1(u)\hat{\pi}_1(v)\hat{\pi}_2(u)\hat{\pi}_2(\varepsilon v) = 1$ for all $u, v \in \mathbb{Z}^2$, and hence the right-hand side of equation (10.23) is equal to 1. Suppose that there exist $u, v \in \mathbb{Z}^2$ such that $\hat{\pi}_1(u+v)\hat{\pi}_2(u+\varepsilon v) \neq 1$. This means that either $u + v \in H$, $u + \varepsilon v \notin H$ or $u + v \notin H$, $u + \varepsilon v \in H$. For each of these cases we get that $(I - \varepsilon)v \notin H$, contrary to the definition of H. Thus the left-hand side of equation (10.23) is also equal to 1. Hence the functions $\hat{\pi}_j(y)$ satisfy equation (10.23).

Set $g_j(y) = (\hat{\nu}_j(y))^k \hat{\pi}_j(y)$. The functions $g_j(y)$ also satisfy equation (10.23). Take k so large that

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} g_j(m,n) < 1.$$

Since $\det A = \det B > 0$, we always can do it. This implies that

$$\rho_j(z,w) = 1 + \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} g_j(m,n)\overline{z^m w^n} > 0, \quad (z,w) \in \mathbb{T}^2.$$

Also it is obvious that

$$\int_{\mathbb{T}^2} \rho_j(z,w)\, dm_{\mathbb{T}^2}(z,w) = 1.$$

Let μ_j be the distributions on $\mathbb{T}^2$ with the densities $\rho_j(z,w)$ with respect to the Haar distribution $m_{\mathbb{T}^2}$. It is obvious that $\hat{\mu}_j(y) = g_j(y)$ and $\mu_j \notin \Gamma(\mathbb{T}^2)$. Let ξ_1 and ξ_2 be independent random variables with values in the group $\mathbb{T}^2$ and distributions μ_j. Since the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (10.23), by Lemma 10.1 the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Statement (IIb) is proved. The theorem is completely proved. □

Remark 11.6. Assume that $\lambda_1\lambda_2 = -1$, $\lambda_1 + \lambda_2 \neq 0$. By Lemma 11.2 there exist nonzero solutions A and B of equation 11.2 (i). Let μ_1 and μ_2 be Gaussian distributions on the group $\mathbb{T}^2$ with the characteristic functions

$$\hat{\mu}_1(y) = \exp\{-\langle Ay, y\rangle\}, \quad \hat{\mu}_2(y) = \exp\{-\langle By, y\rangle\}, \quad y = (m, n) \in \mathbb{Z}^2.$$

Obviously, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (10.23). By Lemma 10.1 it follows from this that if the conditions of statement (Ib) are satisfied there exist non-degenerate Gaussian distributions μ_1 and μ_2 concentrated on shifts of a one-parameter subgroup V of $\mathbb{T}^2$ such that V is dense in $\mathbb{T}^2$, and if ξ_1 and ξ_2 are independent random variables with values in $\mathbb{T}^2$ and distributions μ_1 and μ_2, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent.

Assume that either $\lambda_1 = \lambda_2 = -1$ and $\delta \neq -I$ or $\lambda_1 = 1$, $\lambda_2 = -1$. By Lemma 11.2 there exist nonzero solutions A and B of equation 11.2 (i). Put $K = A(\mathbb{T}^2, M)$, where $M = \operatorname{Ker} A \cap \mathbb{Z}^2 \neq \{0\}$. Then $K \cong \mathbb{T}$ and the restriction of the automorphism δ to the subgroup K coincides with the automorphism $-I$. Let γ be a Gaussian distribution on the group $\mathbb{T}^2$ with the characteristic function

$$\hat{\gamma}(y) = \exp\{-\langle Ay, y\rangle\}, \quad y = (m, n) \in \mathbb{Z}^2.$$

Then the characteristic functions $\hat{\gamma}_j(y) = \hat{\gamma}(y)$, $j = 1, 2$, satisfy equation (10.23). Let F be the subgroup of K generated by the element of order 2. Take $0 < r < 1$ and consider the distribution $\pi_1 = rE_{(1,1)} + (1-r)m_F$ and the signed measure $\pi_2 = (1/r)E_{(1,1)} + (1 - 1/r)m_F$. It is easy to see that $\pi_1 * \pi_2 = E_{(1,1)}$, and the characteristic functions $\hat{\pi}_j(y)$ satisfy equation (10.23). Set $\mu_j = \gamma * \pi_j$, $j = 1, 2$. It is clear that for a given matrix A we can choose a number r in such a manner that μ_j are distributions. The characteristic functions of the distributions μ_j also satisfy equation (10.23). Taking into account Lemma 10.1, it follows from this that if the conditions of statement (IIa) are satisfied there exist non-Gaussian distributions μ_j of the form $\mu_j = \gamma * \pi_j$ which have the following property: If ξ_1 and ξ_2 are independent random variables with values in $\mathbb{T}^2$ and distributions μ_j, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent.

From what has been said it follows that statements (Ib), (IIa) in Theorem 11.5 may not be strengthened. Namely, the following statement is valid.

Let $\delta \in \operatorname{Aut}(\mathbb{T}^2)$ be such a topological automorphism of the group $\mathbb{T}^2$ as in one of the cases (Ib) or (IIa). Then there exist independent random variables ξ_j with values in $\mathbb{T}^2$ and distributions μ_j which are possible by Theorem 11.5 in the corresponding case and such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Thus the description of the classes of distributions which are characterized by the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ in cases (Ib) and (IIa) is sharp.

Remark 11.7. As has been shown in the proof of statement (Ib) of Theorem 11.5, the distributions μ_1 and μ_2 are concentrated on shifts of a one-parameter subgroup V of the form $V = p(\mathbb{R})$. Using the representation for the matrix A obtained in the proof of Lemma 11.2 (see 2A, 2B) it is not difficult to verify that $\delta(V) = V$. We will prove this statement for the case 2A (i). The other cases can be considered similarly. Put $t_0 = (bt_1 - c)/(a - d)$, where t_1 is defined by (11.25). Then the matrix A is of the form

$$A = \sigma \begin{pmatrix} t_1 & t_0 \\ t_0 & 1 \end{pmatrix}, \quad \sigma > 0.$$

Since $\det A = 0$, we have $t_0^2 = t_1$. We recall that $p = \tilde{\tau}$, where $\tau \colon \mathbb{Z}^2 \mapsto \mathbb{R}$ is defined by $\tau(m, n) = t_0 m + n$. We will find p. Let $ps = (z, w)$ and hence $(ps, (m, n)) = z^m w^n$, $s \in \mathbb{R}$, $(m, n) \in \mathbb{Z}^2$. On the other hand $(ps, (m, n)) = (s, \tau(m, n)) = (s, t_0 m + n) = e^{is(t_0 m + n)} = e^{ist_0 m} e^{isn}$. Thus $z^m w^n = e^{ist_0 m} e^{isn}$ for all $(m, n) \in \mathbb{Z}^2$. Hence $ps = (e^{ist_0}, e^{is})$. It follows from this that $\delta(ps) = (e^{is(t_0 a + c)}, e^{is(t_0 b + d)})$. If we verify that $(t_0 a + c)/(t_0 b + d) = t_0$, then the inclusion $\delta(p(\mathbb{R}) \subset \mathbb{R}$ will be proved. We have

$$t_0(t_0 b + d) = t_0^2 b + t_0 d = bt_1 + d\,\frac{bt_1 - c}{a - d} = \frac{abt_1 - dc + ac - ac}{a - d} = t_0 a + c.$$

Thus $\delta(p(\mathbb{R}) \subset \mathbb{R}$, and hence $\delta(p(\mathbb{R}) = \mathbb{R}$. Moreover it is easy to see that $p^{-1}\delta p(s) = (t_0 b + d)s$.

We see that if conditions (Ib) are satisfied, then the assertion that the distributions μ_j are Gaussian can be done directly from the Skitovich–Darmois theorem.

12 Random variables with values in the groups $\mathbb{R} \times \mathbb{T}$ and $\boldsymbol{\Sigma_a} \times \mathbb{T}$

Denote by X either the group $\mathbb{R} \times \mathbb{T}$ or the group $\Sigma_{\boldsymbol{a}} \times \mathbb{T}$. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. Assume that ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ and suppose that L_1 and L_2 are independent. In this section we describe the possible distributions μ_j.

12.1 Notation. Let $X = \mathbb{R} \times \mathbb{T}$. Denote by $x = (t, z)$, $t \in \mathbb{R}$, $z \in \mathbb{T}$, elements of the group X. We have $Y \cong \mathbb{R} \times \mathbb{Z}$. To avoid introducing new notation we will assume that $Y = \mathbb{R} \times \mathbb{Z}$. Denote by $y = (s, n)$, $s \in \mathbb{R}$, $n \in \mathbb{Z}$, elements of Y. For convenience we also assume that the real line $\mathbb{R}$, the circle group $\mathbb{T}$, and the group $\mathbb{Z}(2)$ are embedded in the natural way in the group X. It is easy to verify that every topological automorphism $\varepsilon \in \mathrm{Aut}(\mathbb{R} \times \mathbb{Z})$ is defined by a matrix $\left(\begin{smallmatrix} a & b \\ 0 & \pm 1 \end{smallmatrix}\right)$, where $a, b \in \mathbb{R}$, $a \neq 0$, and ε operates on $\mathbb{R} \times \mathbb{Z}$ as follows:

$$\varepsilon(s, n) = (as + bn, \pm n), \quad (s, n) \in \mathbb{R} \times \mathbb{Z}.$$

Then the adjoint automorphism $\delta = \tilde{\varepsilon} \in \mathrm{Aut}(\mathbb{R} \times \mathbb{T})$ is of the form

$$\delta(t, z) = (at, e^{ibt} z^{\pm 1}), \quad (t, z) \in \mathbb{R} \times \mathbb{T}.$$

We identify δ and ε with the corresponding matrix $\left(\begin{smallmatrix} a & b \\ 0 & \pm 1 \end{smallmatrix}\right)$.

Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. As noted at the beginning of the proof of Theorem 10.11, the study of possible distributions μ_j provided that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent is reduced to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$.

In what follows we need two lemmas.

Lemma 12.2. *Let $\varepsilon = \left(\begin{smallmatrix} a & b \\ 0 & -1 \end{smallmatrix}\right)$, where $a \neq 0$. Let $A = (a_{ij})_{i,j=1}^2$ and $B = (b_{ij})_{i,j=1}^2$ be symmetric positive semidefinite matrices. If either $a > 0$ or $a = -1$ and $b \neq 0$, then all nonzero solutions of the equation*

$$\text{(i)} \quad A + B\varepsilon = 0$$

are of the form

$$\text{(ii)} \quad A = B = \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma > 0.$$

If $a < 0$, $a \neq -1$, then all nonzero solutions of equation (i) *are of the form*

$$\text{(iii)} \quad B = \sigma \begin{pmatrix} 1 & b/(a+1) \\ b/(a+1) & t \end{pmatrix}, \quad A = -B\varepsilon, \quad \sigma > 0,\ t \geq b^2/(a+1)^2.$$

Proof. First note that equation (i) has a nonzero solution if and only if the following system of equations and inequalities has a nonzero solution:

$$\begin{cases} b_{11} \geq 0, \quad b_{22} \geq 0, & (12.1) \\ b_{11}b_{22} - b_{12}^2 \geq 0, & (12.2) \\ b_{11}b - b_{12} = b_{12}a, & (12.3) \\ b_{11}a \leq 0, & (12.4) \\ b_{12}b - b_{22} \leq 0, & (12.5) \\ b_{11}a(b_{12}b - b_{22}) - (b_{12}a)^2 \geq 0. & (12.6) \end{cases}$$

Let $a > 0$. We conclude from (12.1) and (12.4) that $b_{11} = 0$. Then we obtain from (12.2) that $b_{12} = 0$. Obviously, all nonzero solutions of equation (i) can be represented in the form (ii).

Let $a = -1$ and $b \neq 0$. We get from (12.3) that $b_{11} = 0$. As noted above, all nonzero solutions of equation (i) can be represented in the form (ii).

Note that if $a < 0$, then $\det \varepsilon > 0$. It follows from (i) that inequalities (12.2) and (12.6) are equivalent. Let $a < 0$, $a \neq -1$. We obtain from (12.3) that

$$b_{12} = b_{11}b/(a+1). \qquad (12.7)$$

Substituting (12.7) into (12.2) we find that

$$b_{11}b_{22} - b_{11}^2 b^2/(a+1)^2 \geq 0. \tag{12.8}$$

As noted above, if $b_{11} = 0$, then all nonzero solutions of equation (i) can be represented in the form (ii). Let $b_{11} = \sigma > 0$. Then (12.8) is equivalent to the inequality

$$b_{22} - b_{11}b^2/(a+1)^2 \geq 0. \tag{12.9}$$

Substituting (12.7) into (12.5) we find that

$$b_{22} - b_{11}b^2/(a+1) \geq 0. \tag{12.10}$$

Put $t = b_{22}/b_{11}$. Then inequalities (12.9) and (12.10) become

$$t \geq b^2/(a+1)^2, \quad t \geq b^2/(a+1). \tag{12.11}$$

Obviously, if $a < 0$, $a \neq -1$, then system of inequalities (12.11) is equivalent to the inequality

$$t \geq b^2/(a+1)^2.$$

It follows from this that all nonzero solutions of equation (i) can be represented in the form (iii). □

Lemma 12.3. *Let Y be an arbitrary Abelian group, ε be an automorphism of the group Y. Assume that normalized functions $f_j(y)$ do not vanish and satisfy the equation*

(i) $\quad f_1(u+v)f_2(u+\varepsilon v) = f_1(u)f_1(v)f_2(u)f_2(\varepsilon v), \quad u,\ v \in Y.$

Then every function $f_j(y)$ satisfies the equation

(ii) $\quad f_j(u+v)f_j(u-v) = f_j^2(u)f_j(v)f_j(-v), \quad u \in (\varepsilon - I)Y,\ v \in Y.$

Proof. Let p and q be arbitrary elements of the group Y. Putting $u = p - q$, $v = q$ in equation (i) we obtain

$$f_1(p)f_2(p + (\varepsilon - I)q) = f_1(p-q)f_1(q)f_2(p-q)f_2(\varepsilon q), \quad p, q \in Y. \tag{12.12}$$

Let h be an arbitrary element of the group Y. Substituting $p + h$ for p and $q + h$ for q into equation (12.12) we infer that

$$\begin{aligned} &f_1(p+h)f_2(p + (\varepsilon - I)q + \varepsilon h) \\ &\quad = f_1(p-q)f_1(q+h)f_2(p-q)f_2(\varepsilon q + \varepsilon h), \quad p, q, h \in Y. \end{aligned} \tag{12.13}$$

Dividing equation (12.13) by equation (12.12) we find

$$\frac{f_1(p+h)f_2(p + (\varepsilon - I)q + \varepsilon h)}{f_1(p)f_2(p + (\varepsilon - I)q)} = \frac{f_1(q+h)f_2(\varepsilon q + \varepsilon h)}{f_1(q)f_2(\varepsilon q)}, \quad p, q, h \in Y. \tag{12.14}$$

Let k be an arbitrary element of the group Y. Substituting $p + (\varepsilon - I)k$ for p and $q - k$ for q into equation (12.14) we have

$$\begin{aligned}\frac{f_1(p+h+(\varepsilon-I)k)f_2(p+(\varepsilon-I)q+\varepsilon h)}{f_1(p+(\varepsilon-I)k)f_2(p+(\varepsilon-I)q)}\\ =\frac{f_1(q+h-k)f_2(\varepsilon q+\varepsilon h-\varepsilon k)}{f_1(q-k)f_2(\varepsilon q-\varepsilon k)}, \quad p,q,h,k\in Y.\end{aligned} \tag{12.15}$$

Dividing equation (12.15) by equation (12.14) we find

$$\begin{aligned}\frac{f_1(p)f_1(p+h+(\varepsilon-I)k)}{f_1(p+h)f_1(p+(\varepsilon-I)k)}\\ =\frac{f_1(q+h-k)f_1(q)f_2(\varepsilon q)f_2(\varepsilon q+\varepsilon h-\varepsilon k)}{f_1(q-k)f_1(q+h)f_2(\varepsilon q+\varepsilon h)f_2(\varepsilon q-\varepsilon k)}, \quad p,q,h,k\in Y.\end{aligned} \tag{12.16}$$

The right-hand side of equation (12.16) does not depend on p. Hence the left-hand side of equation (12.16) takes the same value for $p = -h$ and for $p = 0$. We obtain

$$\frac{f_1(-h)f_1((\varepsilon-I)k)}{f_1(-h+(\varepsilon-I)k)}=\frac{f_1(h+(\varepsilon-I)k)}{f_1(h)f_1((\varepsilon-I)k)}, \quad h,k\in Y. \tag{12.17}$$

It follows from this that

$$f_1((\varepsilon-I)k+h)f_1((\varepsilon-I)k-h)=f_1^2((\varepsilon-I)k)f_1(h)f_1(-h), \quad h,k\in Y. \tag{12.18}$$

Putting $u = (\varepsilon - I)k$, $v = h$ in equation (12.18) we get that the function $f_1(y)$ satisfies equation (ii).

Set $v' = \varepsilon v$. Then equation (i) becomes

$$f_1(u+\varepsilon^{-1}v')f_2(u+v')=f_1(u)f_1(\varepsilon^{-1}v')f_2(u)f_2(v'), \quad u\in(\varepsilon-I)Y,\ v\in Y.$$

Since $(\varepsilon^{-1} - I)Y = (\varepsilon - I)Y$, the reasoning given above shows that the function $f_2(y)$ also satisfies equation (ii). □

Theorem 12.4. *Let $X = \mathbb{R}\times\mathbb{T}$, $\delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with the non-vanishing characteristic functions. Suppose that linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Then the following statements hold:*

(a) *If $\delta = \begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$, where $a > 0$, then μ_j are degenerate distributions.*

(b) *If $\delta = \begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$, where $a < 0$, then either μ_j are degenerate distributions or $\mu_j = E_{x_j} * \gamma_j$, where $\gamma_j \in \Gamma(X)$, $\sigma(\gamma_j) = K$, K is a subgroup of X topologically isomorphic to $\mathbb{R}$.*

(c) *If either $\delta = \begin{pmatrix} a & b \\ 0 & -1\end{pmatrix}$, where $a > 0$, or $\delta = \begin{pmatrix} -1 & b \\ 0 & -1\end{pmatrix}$, where $b \neq 0$, then either μ_j are degenerate distributions or $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in X$, $\gamma \in \Gamma(X)$, $\sigma(\gamma) = \mathbb{T}$, π_j are signed measures on $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$.*

(d) *If* $\delta = \left(\begin{smallmatrix} a & b \\ 0 & -1\end{smallmatrix}\right)$, *where* $a < 0$, $a \neq -1$, *then either* $\mu_j = \gamma_j * \pi_j$, *where* $\gamma_j \in \Gamma(X)$, $\sigma(\gamma_j) = X$, π_j *are signed measures on* $\mathbb{Z}(2)$ *such that* $\pi_1 * \pi_2 = E_{(0,1)}$, *or* μ_j *are the same as in* (b).

(e) *If* $\delta = -I$, *then either* μ_j *are the same as in* (c) *or* μ_j *are the same as in* (d).

Proof. Put $\varepsilon = \tilde{\delta}$, $L = \mathrm{Ker}(\varepsilon - I)$, $H = (\varepsilon - I)Y$. By Lemma 10.1 the independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.23). Two cases are possible: $L \neq \{0\}$ and $L = \{0\}$.

1. $L \neq \{0\}$. Then either $\delta = \left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)$ or $\delta = \left(\begin{smallmatrix} 1 & b \\ 0 & -1\end{smallmatrix}\right)$. Put $K = A(X, L)$. It is obvious that the restrictions of the characteristic functions $\hat{\mu}_1(y)$ and $\hat{\mu}_2(y)$ to the subgroup L satisfy equation (11.28). Reasoning as in case 1 of Theorem 11.5 we conclude that the distributions μ_j can be replaced by their shifts μ_j' in such a way that the supports $\sigma(\mu_j') \subset K$, $j = 1, 2$. Moreover, the restriction of the automorphism δ to the subgroup K is an automorphism of the subgroup K. Taking into account Corollary 10.2 we can assume from the beginning that the random variables ξ_j take values in the subgroup K.

We will first prove statements (a) and (b). Let $\delta = \left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)$. It is obvious that in the case when $\delta = I$, μ_j are degenerate distributions. Let $\delta = \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)$, $b \neq 0$. Then $L = \{(s, 0) : s \in \mathbb{R}\}$, and hence $K = \mathbb{T}$. It is clear that the restriction of the automorphism δ to the subgroup $\mathbb{T}$ coincides with the identity automorphism I. Thus we have independent random variables ξ_1 and ξ_2 with values in $\mathbb{T}$ and distributions μ_1 and μ_2 such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \xi_2$ are independent. It follows from this that μ_j are degenerate distributions.

Let $\delta = \left(\begin{smallmatrix} a & b \\ 0 & 1\end{smallmatrix}\right)$, where $a \neq 1$. It is easy to verify that $L = \{(\frac{b}{1-a}n, n) : n \in \mathbb{Z}\}$. This implies that $K = \{(t, e^{it\frac{b}{a-1}}) : t \in \mathbb{R}\}$, i.e., the subgroup K is topologically isomorphic to $\mathbb{R}$. Consider the monomorphism $\tau\colon \mathbb{R} \mapsto X$ of the form $\tau t = (t, e^{it\frac{b}{a-1}})$ and the independent random variables $\tilde{\xi}_j = \tau^{-1}(\xi_j)$. Since $\delta(t, e^{it\frac{b}{a-1}}) = (at, e^{iat\frac{b}{a-1}})$, we get $\tau^{-1}\delta(\xi_2) = a\tilde{\xi}_2$. Thus we have the independent random variables $\tilde{\xi}_j$ taking values in the group $\mathbb{R}$ such that the linear forms $\tilde{L}_1 = \tilde{\xi}_1 + \tilde{\xi}_2$ and $\tilde{L}_2 = \tilde{\xi}_1 + a\tilde{\xi}_2$ are independent. By the Skitovich–Darmois theorem the random variables $\tilde{\xi}_j$ have Gaussian distributions, and hence ξ_j also have Gaussian distributions, i.e., $\mu_j \in \Gamma(X)$.

Obviously, if $a > 0$, then μ_j are degenerate distributions. If $a < 0$, then there exist independent random variables ξ_j with values in K and nondegenerate Gaussian distributions μ_j such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Statements (a) and (b) are proved.

Let $\delta = \left(\begin{smallmatrix} 1 & b \\ 0 & -1\end{smallmatrix}\right)$. Then $L = \{(s, 0) : s \in \mathbb{R}\}$, and hence $K = \mathbb{T}$. Obviously, the restriction of the automorphism δ to the subgroup $\mathbb{T}$ coincides with the automorphism $-I$. By Corollary 8.6, $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in X$, $\gamma \in \Gamma(X)$, $\sigma(\gamma) = \mathbb{T}$, π_j are signed measures on $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$. Statement (c) in the case where $a = 1$ is proved.

2. We now consider the case that $L = \{0\}$. Then $\delta = \left(\begin{smallmatrix} a & b \\ 0 & -1\end{smallmatrix}\right)$, where $a \neq 1$. We have $H = (I - \varepsilon)Y = Y^{(2)} = \mathbb{R} \times \mathbb{Z}^{(2)}$. Put $G = H^*$. Since the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (10.23), it follows from Lemma 12.3 that every characteristic function $\hat{\mu}_j(y)$ satisfies equation 12.3 (ii) on the subgroup $Y^{(2)}$. Therefore by Lemma 9.2 these restrictions are the characteristic functions of Gaussian distributions in the sense of Bernstein on the group G. Since $H \cong \mathbb{R} \times \mathbb{Z}$, Theorem 1.7.1 implies that $G \cong \mathbb{R} \times \mathbb{T}$. The group G contains no subgroup topologically isomorphic to the two-dimensional torus $\mathbb{T}^2$. Therefore, applying Lemma 9.7 to the group G we obtain that the restrictions of the characteristic functions $\hat{\mu}_j(y)$ to the subgroup H are the characteristic functions of Gaussian distributions. It follows from Remark 5.12 that a continuous function $\varphi(y)$ on the group $\mathbb{R} \times \mathbb{Z}$ satisfying equation 2.16 (ii) is of the form $\varphi(y) = \exp\{-\langle Cy, y\rangle\}$, where C is a symmetric positive semidefinite matrix. Hence we obtain the following representation for the restrictions of the characteristic functions $\hat{\mu}_j(y)$ to the subgroup H

$$\hat{\mu}_1(y) = m_1(y)\exp\{-\langle Ay, y\rangle\}, \quad \hat{\mu}_2(y) = m_2(y)\exp\{-\langle By, y\rangle\}, \quad u,\ v \in H,$$

where $m_j(y)$ is a character of the subgroup H, and A, B are symmetric positive semidefinite matrices. By Theorem 1.9.2 there exist elements $x_j \in X$ such that $m_j(y) = (x_j, y)$, $y \in H$. Put $\mu_j' = \mu_j * E_{-x_j}$. Taking into account Corollary 10.2, we can assume from the beginning that the following representations hold:

$$\hat{\mu}_1(y) = \exp\{-\langle Ay, y\rangle\}, \quad \hat{\mu}_2(y) = \exp\{-\langle By, y\rangle\}, \quad y \in H. \tag{12.19}$$

Consider the restriction of equation (10.23) to the subgroup H and substitute the expressions (12.19) for $\hat{\mu}_j(y)$ in (10.23). We obtain that

$$\langle u, (A + B\varepsilon)v\rangle = 0, \quad u, v \in H. \tag{12.20}$$

Since $H \cong \mathbb{R} \times \mathbb{Z}$, it follows from (12.20) that the matrices A and B satisfy equation 12.2 (i).

Put

$$l_j(y) = \hat{\mu}_j(y)/|\hat{\mu}_j(y)|, \quad y \in Y,$$

and verify that the functions $l_j(y)$ are characters of the group Y. Since $\hat{\mu}_j(y) = |\hat{\mu}_j(y)|$, $y \in H$, we have $l_j(y) = 1$, $y \in H$. The functions $l_j(y)$ satisfy equation (10.23). By Lemma 12.3 every function $l_j(y)$ satisfies equation 12.3 (ii) which takes the form

$$l_j(u + v)l_j(u - v) = 1, \quad u \in H,\ v \in Y. \tag{12.21}$$

Substituting $u = (s/2, 0)$, $v = (s/2, n)$ into (12.21) we find that

$$l_j(s, n)l_j(0, -n) = 1, \quad s \in \mathbb{R},\ n \in \mathbb{Z}.$$

Multiplying this equation by $l_j(0, n)$, we arrive at

$$l_j(s, n) = l_j(0, n), \quad s \in \mathbb{R},\ n \in \mathbb{Z}. \tag{12.22}$$

In view of (12.22) it follows from equation (10.23) for the functions $l_j(y)$ that the functions $l_j(y)$ satisfy the equation

$$l_1(0, m + n)l_2(0, m - n) = l_1(0, m)l_1(0, n)l_2(0, m)l_2(0, -n), \quad m, n \in \mathbb{Z}. \tag{12.23}$$

Taking into account equation (12.23) it is easy to verify by induction that the functions $l_j(0,n)$ are characters of the group $\mathbb{Z}$. Hence $l_j(y)$ are characters of the group Y. By the Pontryagin duality theorem there exist elements $x_j\in X$ such that

$$l_j(y)=(x_j,y),\quad y\in Y. \tag{12.24}$$

Now we will find representations for the modules $|\hat\mu_j(y)|$. To this aim, put $\varphi_j(y)=-\ln|\hat\mu_j(y)|$. We conclude from (10.23) that the functions $\varphi_j(y)$ satisfy equation 10.9 (i). By Lemma 10.9 the function $\varphi_j(y)$ satisfies equation 10.9 (ii). Since $(I-\varepsilon)Y=Y^{(2)}$, it follows from equation 10.9 (ii) that the function $\varphi_j(y)$ satisfies equation (8.13).

The solution of equation (8.13) on the group $\mathbb{R}\times\mathbb{Z}$ was obtained in the proof of Theorem 8.5, and it was based on the representation of solutions of equation (8.13) on the group $\mathbb{Z}$. Below we solve equation (8.13) in a different way.

Since the function $\varphi_1(y)$ satisfies equation (8.13), the function $\varphi_1(y)$ satisfies the equation

$$\Delta_h^3\varphi_1(y)=0,\quad h,y\in Y^{(2)}, \tag{12.25}$$

i.e., the restriction of the function $\varphi_1(y)$ to the subgroup $Y^{(2)}$ is a polynomial of degree ≤ 2. It follows from (12.19) that

$$\varphi_1(y)=\langle Ay,y\rangle,\quad \varphi_2(y)=\langle By,y\rangle,\quad y\in Y^{(2)}. \tag{12.26}$$

Put $\zeta=(0,1)\in Y$. Represent Y as the union $Y=Y^{(2)}\cup\{\zeta+Y^{(2)}\}$. Consider the function $\tilde\varphi_1(y)=\varphi_1(\zeta+y)$, $y\in Y$. Obviously, the function $\tilde\varphi_1(y)$ also satisfies equation (8.13). Thus its restriction to the subgroup $Y^{(2)}$ is a polynomial of degree ≤ 2. By Theorem 5.5 the function $\tilde\varphi_1(y)$ can be represented in the form 5.5 (i), i.e.,

$$\tilde\varphi_1(y)=g_2(y,y)+g_1(y)+g_0,\quad y\in Y^{(2)}, \tag{12.27}$$

where $g_2(u,v)$ is a 2-additive function, $g_1(y)$ is an additive function and $g_0=\text{const}$. Obviously, the function g_j can be extended from the subgroup $Y^{(2)}$ to Y retaining its properties. Since $\varphi_1(y)=\varphi_1(-y)$, $y\in Y$, we have

$$\begin{aligned}\tilde\varphi_1(y)&=\varphi_1(-\zeta-y)=\varphi_1(\zeta-2\zeta-y)=\tilde\varphi_1(-2\zeta-y)\\&=g_2(-2\zeta-y,-2\zeta-y)+g_1(-2\zeta-y)+g_0\\&=g_2(y,y)+4g_2(\zeta,y)+4g_2(\zeta,\zeta)-g_1(y)-2g_1(\zeta)+g_0,\quad y\in Y^{(2)}.\end{aligned} \tag{12.28}$$

We deduce from (12.27) and (12.28) that

$$g_1(y)=2g_2(y,\zeta),\quad y\in Y^{(2)}. \tag{12.29}$$

Taking into account (12.29) we get

$$\varphi_1(y)=\langle A_1y,y\rangle+c_1,\quad y\in(0,1)+Y^{(2)}, \tag{12.30}$$

where A_1 is a symmetric matrix, $c_1 \in \mathbb{R}$. Set $w = v$ in equation (8.13) and rewrite the resulting equation in the form

$$\varphi_1(u+4v) - 2\varphi_1(u+3v) + 2\varphi_1(u+v) - \varphi_1(u) = 0, \quad u, v \in Y.$$

Putting here $u \in Y^{(2)}$, $v \in (0,1) + Y^{(2)}$ and using representations (12.26) and (12.30) we find

$$\langle (A - A_1)u, v\rangle + 2\langle (A - A_1)v, v\rangle, \quad u \in Y^{(2)},\ v \in (0,1) + Y^{(2)}.$$

It follows from this that $A_1 = A$. Thus

$$\varphi_1(y) = \langle Ay, y\rangle + c_1, \quad y \in (0,1) + Y^{(2)}. \tag{12.31}$$

Arguing as above we obtain the representation

$$\varphi_2(y) = \langle By, y\rangle + c_2, \quad y \in (0,1) + Y^{(2)}. \tag{12.32}$$

Substitute $u, v \in (0,1) + Y^{(2)}$ in 10.9 (i). Combining representations (12.26), (12.31), and (12.32), we conclude that $c_1 = -c_2$. Set $c_1 = -c_2 = 2\kappa$. Thus the functions $|\hat{\mu}_j(y)|$ are of the form

$$|\hat{\mu}_1(y)| = \exp\{-\langle Ay, y\rangle + \kappa(1 - (-1))^n\}, \quad y = (s, n) \in Y,$$
$$|\hat{\mu}_2(y)| = \exp\{-\langle By, y\rangle - \kappa(1 - (-1))^n\}, \quad y = (s, n) \in Y.$$

Taking into account (12.24), we finally obtain

$$\hat{\mu}_1(y) = (x_1, y)\exp\{-\langle Ay, y\rangle + \kappa(1 - (-1))^n\}, \quad y = (s, n) \in Y, \tag{12.33}$$
$$\hat{\mu}_2(y) = (x_2, y)\exp\{-\langle By, y\rangle - \kappa(1 - (-1))^n\}, \quad y = (s, n) \in Y. \tag{12.34}$$

As has been shown above, the matrices A and B satisfy equation 12.2 (i). Let either $\delta = \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}$, where $a > 0$, $a \neq 1$, or $\delta = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, where $b \neq 0$. By Lemma 12.2 matrices A and B are of the form 12.2 (ii). Let γ be a Gaussian distribution on the group X with the characteristic function $\hat{\gamma}(y) = \exp\{-\sigma n^2\}$, $y = (s, n) \in Y$. Since $\{y \in Y : \hat{\gamma}(y) = 1\} = \mathbb{R}$ and $A(X, \mathbb{R}) = \mathbb{T}$, it follows from Proposition 2.13 that $\sigma(\gamma) = \mathbb{T}$. Taking into account 2.7 (b) and 2.7 (c), the desired representations for the distributions μ_j follow from (12.33), (12.34), and Lemma 8.3. Since the case $a = 1$ has been already studied earlier, statement (c) is completely proved.

Let $\delta = \begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix}$, where $a < 0$, $a \neq -1$. By Lemma 12.2 matrices A and B are of the form 12.2 (iii). Let γ_j be Gaussian distributions on the group X with the characteristic functions

$$\hat{\gamma}_1(y) = (x_1, y)\exp\{-\langle Ay, y\rangle\}, \quad \hat{\gamma}_2(y) = (x_2, y)\exp\{-\langle By, y\rangle\}, \quad y \in Y.$$

Taking into account 2.7 (b) and 2.7 (c), we deduce from (12.33) and Lemma 8.3 that $\mu_1 = \gamma_1 * \pi_1$. If $\sigma > 0$ and $t > b^2/(a+1)^2$, then $\det A > 0$. Therefore $\sigma(\gamma_1) = X$.

Arguing as above we obtain from (12.34) and Lemma 8.3 the desired representation for the distribution μ_2.

If $\sigma > 0$ and $t = b^2/(a+1)^2$, then $A = -aB$ and $\det A = \det B = 0$. Consider $M = \operatorname{Ker} A$. We have $M = \{(-\frac{b}{a+1}n, n) : n \in \mathbb{Z}\}$. Put $K = A(X, M)$. It is easy to see $K = \{(t, e^{it\frac{b}{a+1}}) : t \in \mathbb{R}\}$. Obviously, the subgroup K is topologically isomorphic to the group $\mathbb{R}$. By Proposition 2.13 the inclusions $\sigma(\gamma_j) \subset A(X, M)$, $j = 1, 2$, are valid. Since $\gamma \in \Gamma(X)$, we have $\sigma(\gamma) = K$. It follows from this that in (12.33) and (12.34) $\kappa = 0$. In the opposite case one of the distributions μ_j is a signed measure rather than a measure. Hence $\mu_j = E_{x_j} * \gamma_j$. Statement (d) is proved.

Let $\delta = -I$. Then $A = B$ and the desired representations for μ_j follow from (12.33), (12.34), and Lemma 8.3. Statement (e) is proved. □

Remark 12.5. It should be noted that statements (b)–(e) in Theorem 12.4 may not be strengthened. Namely, the following statement is valid.

Let $\delta \in \operatorname{Aut}(X)$ be such a topological automorphism of the group X as in one of cases (b)–(e). Then there exist independent random variables ξ_j with values in X and distributions μ_j which are possible by Theorem 12.4 in the corresponding case and such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Thus the description of the classes of distributions which are characterized by the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ in cases (b)–(e) is sharp.

Now we come to the study of distributions on the group $X = \Sigma_{\boldsymbol{a}} \times \mathbb{T}$ which are characterized by the independence of linear forms.

12.6 Notation. Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_n, \ldots)$ be a fixed but arbitrary infinite sequence of integers, where each of a_n is greater than 1, let $\Sigma_{\boldsymbol{a}}$ be the $\boldsymbol{a}$-adic solenoid (see 1.2 (g)). The character group $\Sigma_{\boldsymbol{a}}^*$ is topologically isomorphic to the subgroup of $\mathbb{Q}$ of the form $H_{\boldsymbol{a}}$, where $H_{\boldsymbol{a}} = \left\{\frac{m}{a_0 a_1 \ldots a_n} : n = 0, 1, \ldots; m \in \mathbb{Z}\right\}$ (see 1.10 (e)). Let $X = \Sigma_{\boldsymbol{a}} \times \mathbb{T}$. Then $Y = X^* \cong H_{\boldsymbol{a}} \times \mathbb{Z}$. To avoid introducing new notation we will assume that $Y = H_{\boldsymbol{a}} \times \mathbb{Z}$. Denote by $x = (g, z)$, $g \in \Sigma_{\boldsymbol{a}}$, $z \in \mathbb{T}$, elements of the group X and by $y = (r, n)$, $r \in H_{\boldsymbol{a}}$, $n \in \mathbb{Z}$, elements of the group Y. For convenience we also suppose that the $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$, the circle group $\mathbb{T}$, and the group $\mathbb{Z}(2) \subset \mathbb{T}$ are embedded in the natural way in the group X, and the group $H_{\boldsymbol{a}}$ is embedded in the natural way in the group Y.

Every automorphism $a \in \operatorname{Aut}(H_{\boldsymbol{a}})$ is of the form $a = f_p f_q^{-1}$, where $f_p, f_q \in \operatorname{Aut}(H_{\boldsymbol{a}})$, i.e., a operates in $H_{\boldsymbol{a}}$ as multiplication by the rational number $\frac{p}{q}$ (1.14 (d)). Obviously, $\frac{p}{q} \in H_{\boldsymbol{a}}$. We identify a with the rational number $\frac{p}{q}$. In view of 1.13 (d), $\tilde{a} = f_p f_q^{-1} \in \operatorname{Aut}(\Sigma_{\boldsymbol{a}})$. We also identify the automorphism $\tilde{a}$ with the rational number $\frac{p}{q} \in H_{\boldsymbol{a}}$.

It is easy to show that every automorphism $\varepsilon \in \operatorname{Aut}(H_{\boldsymbol{a}} \times \mathbb{Z})$ is defined by a matrix $\begin{pmatrix} a & b \\ 0 & \pm 1 \end{pmatrix}$, where $a \in \operatorname{Aut}(H_{\boldsymbol{a}})$, $b \in H_{\boldsymbol{a}}$, and ε operates on $H_{\boldsymbol{a}} \times \mathbb{Z}$ as follows:

$$\varepsilon(r, n) = (ar + bn, \pm n) \quad (r, n) \in H_{\boldsymbol{a}} \times \mathbb{Z}.$$

Then the adjoint automorphism $\tilde{\varepsilon} = \delta \in \mathrm{Aut}(\Sigma_{\boldsymbol{a}} \times \mathbb{T})$ is of the form

$$\delta(g, z) = (\tilde{a}g, (g, b)z^{\pm 1}), \quad (g, z) \in \Sigma_{\boldsymbol{a}} \times \mathbb{T}.$$

We identify δ and ε with the corresponding matrix $\left(\begin{smallmatrix} a & b \\ 0 & \pm 1 \end{smallmatrix}\right)$.

Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. As noted at the beginning of the proof of Theorem 10.11, the study of possible distributions μ_j provided that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent is reduced to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$.

12.7. Let $X = \Sigma_{\boldsymbol{a}} \times \mathbb{T}$, $Y = H_{\boldsymbol{a}} \times \mathbb{Z}$. Denote by ι the natural embedding $\iota\colon Y \mapsto \mathbb{R} \times \mathbb{Z}$, $\iota(r, n) = (r, n)$. Let $\tau = \tilde{\iota}$ be the adjoint homomorphism $\tau\colon \mathbb{R} \times \mathbb{T} \mapsto X$. Put $g_t = \tau(t, 1)$, $g_t \in \Sigma_{\boldsymbol{a}}$. Then $\tau(t, z) = (g_t, z)$. Obviously, $(g_t, r) = e^{itr}$, $t \in \mathbb{R}$, $r \in H_{\boldsymbol{a}}$. Since $\overline{\iota(Y)} = \mathbb{R} \times \mathbb{Z}$ and ι is a monomorphism, in view of 1.13 (b) τ is a monomorphism and $\overline{\tau(\mathbb{R} \times \mathbb{T})} = X$.

Lemma 12.8. *Let a be a rational number, and let $\tau\colon \mathbb{R}\times\mathbb{T} \mapsto X$ be the homomorphism defined in 12.7. Let $K = \{(t, e^{ita}) : t \in \mathbb{R}\}$ be the subgroup of $\mathbb{R} \times \mathbb{T}$. Then $\overline{\tau(K)} \cong \Sigma_{\boldsymbol{a}}$.*

Proof. Put $L = A(Y, \overline{\tau(K)})$. Since $\tau(K) = \{(g_t, e^{ita}) : t \in \mathbb{R}\}$, we have $L = \{(r, n) \in Y : r + an = 0\}$. Obviously, $L \cong \mathbb{Z}$. Let (r_0, n_0) be a generator of the subgroup L. Then $L = \{k(r_0, n_0) : k \in \mathbb{Z}\}$. Consider the homomorphism $\pi : Y \mapsto H_{\boldsymbol{a}}$ of the form $\pi(r, n) = rn_0 - nr_0$. It is easy to see that $\mathrm{Ker}\,\pi = L$. Hence $Y/L \cong \pi(Y)$. We have $H_{\boldsymbol{a}} \cong H_{\boldsymbol{a}}^{(n_0)} \subset \pi(Y) \subset H_{\boldsymbol{a}}$. Since $H_{\boldsymbol{a}}$ is a torsion-free group of rank 1, it follows from this that $\pi(Y) \cong H_{\boldsymbol{a}}$ (see 1.3). Hence $Y/L \cong H_{\boldsymbol{a}}$. By Theorem 1.9.2, $(\overline{\tau(K)})^* \cong Y/L \cong H_{\boldsymbol{a}}$. Applying the Pontryagin duality theorem we conclude that $\overline{\tau(K)} \cong \Sigma_{\boldsymbol{a}}$. □

Theorem 12.9. *Let $X = \Sigma_{\boldsymbol{a}} \times \mathbb{T}$, $\delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Suppose that linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Then the following statements hold:*

(a) *If $\delta = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, where $a > 0$, then μ_j are degenerate distributions.*

(b) *If $\delta = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, where $a < 0$, then either μ_j are degenerate distributions or $\mu_j = E_{x_j} * \gamma_j$, where $\gamma_j \in \Gamma(X)$, $\sigma(\gamma_j) = K$, K is a subgroup of X topologically isomorphic to the group $\Sigma_{\boldsymbol{a}}$.*

(c) *If either $\delta = \left(\begin{smallmatrix} a & b \\ 0 & -1 \end{smallmatrix}\right)$, where $a > 0$, or $\delta = \left(\begin{smallmatrix} -1 & b \\ 0 & -1 \end{smallmatrix}\right)$, where $b \neq 0$, then either μ_j are degenerate distributions or $\mu_j = E_{x_j} * \gamma * \pi_j$, where $x_j \in X$, $\gamma \in \Gamma(X)$, $\sigma(\gamma) = \mathbb{T}$, π_j are signed measures on $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$.*

(d) *If $\delta = \left(\begin{smallmatrix} a & b \\ 0 & -1 \end{smallmatrix}\right)$, where $a < 0$, $a \neq -1$, then either $\mu_j = \gamma_j * \pi_j$, where $\gamma_j \in \Gamma(X)$, $\sigma(\gamma_j) = X$, π_j are signed measures on $\mathbb{Z}(2)$ such that $\pi_1 * \pi_2 = E_{(0,1)}$, or μ_j are the same as in* (b).

(e) *If* $\delta = -I$ *then* μ_j *are the same as in* (c) *or* μ_j *are the same as in* (d).

Proof. Put $\varepsilon = \tilde{\delta}$. By Lemma 10.1 the independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.23). Put $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$, $y \in Y$, and the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.23). Note that the restriction of the automorphism ε to the subgroup $H_{\boldsymbol{a}}$ is an automorphism of the subgroup $H_{\boldsymbol{a}}$. Consider the restriction of equation (10.23) for characteristic functions $\hat{\nu}_j(y)$ to the subgroup $H_{\boldsymbol{a}}$. Since $H_{\boldsymbol{a}}^* \cong \Sigma_{\boldsymbol{a}}$ and the group $\Sigma_{\boldsymbol{a}}$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, by Lemma 10.1 and Theorem 10.3, $\hat{\nu}_j(r,0)$ are the characteristic functions of Gaussian distributions on the group $\Sigma_{\boldsymbol{a}}$. Since $\hat{\nu}_j(y) > 0$, $y \in Y$, we have

$$\hat{\nu}_j(r,0) = \exp\{-\sigma r^2\}, \quad r \in H_{\boldsymbol{a}}, \tag{12.35}$$

where $\sigma \geq 0$. Let ι, τ and g_t be the same as in 12.7. Taking into account inequality 2.7 (g) it follows from (12.35) that the characteristic functions $\hat{\nu}_j(r,n)$, $j = 1,2$, are uniformly continuous on the subgroup $\iota(Y)$ in the topology induced on $\iota(Y)$ by the topology of the group $\mathbb{R}\times\mathbb{Z}$. Since the subgroup $\iota(Y)$ is dense in $\mathbb{R}\times\mathbb{Z}$ in the topology of $\mathbb{R}\times\mathbb{Z}$, the characteristic functions $\hat{\nu}_j(r,n)$ can be extended by continuity to some continuous functions $g_j(s,n)$ on the group $\mathbb{R}\times\mathbb{Z}$. Obviously, the functions $g_j(s,n)$ are positive definite. By the Bochner theorem there exist distributions λ_j on the group $\mathbb{R}\times\mathbb{T}$ such that $\hat{\lambda}_j(s,n) = g_j(s,n)$, $j = 1,2$. It follows from Proposition 2.10 that $\nu_j = \tau(\lambda_j)$. Hence the distributions ν_j are concentrated on the Borel subgroup $B = \tau(\mathbb{R}\times\mathbb{T})$. We verify that B is a characteristic subgroup of the group X. Let $\delta = \left(\begin{smallmatrix} a & b \\ 0 & \pm 1\end{smallmatrix}\right) \in \mathrm{Aut}(X)$ and $x = (g_t, z) \in B$. Then we have $\delta x = \delta(g_t,z) = (ag_t, (g_t,b)z^{\pm 1}) = (g_{at}, e^{ibt}z^{\pm 1}) \in B$, i.e., the subgroup B is invariant with respect to every automorphism $\delta \in \mathrm{Aut}(X)$. By the automorphism δ we define a mapping $\bar{\delta}\colon \mathbb{R}\times\mathbb{T} \mapsto \mathbb{R}\times\mathbb{T}$ as $\bar{\delta}(t,z) = \tau^{-1}\delta\tau(t,z)$. Obviously, $\bar{\delta} \in \mathrm{Aut}(\mathbb{R}\times\mathbb{T})$. Since

$$\bar{\delta}(t,z) = \tau^{-1}\delta\tau(t,z) = \tau^{-1}\delta(g_t,z) = \tau^{-1}(g_{at}, e^{ibt}z^{\pm 1}) = (at, e^{ibt}z^{\pm 1}),$$

the automorphism $\bar{\delta}$ can be also identified with the matrix $\left(\begin{smallmatrix} a & b \\ 0 & \pm 1\end{smallmatrix}\right)$ which corresponds to the automorphism δ.

Since the distributions ν_j are concentrated on the Borel subgroup B and $\nu_j = \mu_j * \bar{\mu}_j$, by Proposition 2.2 the distributions μ_j can be replaced by the shifts μ'_j in such a way that the distributions μ'_j are concentrated on the subgroup B. Taking into account Corollary 10.2, we can assume from the beginning that the distributions μ_j are concentrated on the subgroup B. Since τ is a one-to-one continuous mapping, by the Suslin theorem, images of Borel sets under the mapping τ are also Borel sets.

Consider the independent random variables $\tilde{\xi}_j = \tau^{-1}\xi_j$ taking values in the group $\mathbb{R}\times\mathbb{T}$. Since $\tau^{-1}\delta\xi_2 = \bar{\delta}\tilde{\xi}_2$, the linear forms $\tilde{L}_1 = \tilde{\xi}_1 + \tilde{\xi}_2$ and $\tilde{L}_2 = \tilde{\xi}_1 + \bar{\delta}\tilde{\xi}_2$ are independent. Let $\tilde{\mu}_j$ be the distributions of the random variables $\tilde{\xi}_j$. Since $\mu_j = \tau(\tilde{\mu}_j)$, statements (a)–(e) result from statements (a)–(e) of Theorem 12.4 and Lemma 12.8. □

Remark 12.10. Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. Suppose that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent. As noted in Remark 10.5, we can replace the random variables ξ_j by their shifts ξ_j' in such a way that their distributions μ_j' are supported in the connected component of zero of the group X. Furthermore, by Corollary 10.2 the linear forms $L_1' = \alpha_1\xi_1' + \alpha_2\xi_2'$ and $L_2' = \beta_1\xi_1' + \beta_2\xi_2'$ remain independent. Therefore, without restricting the generality, it suffices to study the possible distributions μ_j assuming that the group X is connected.

One of the main characteristics of a connected locally compact Abelian group X is its dimension. If X is a group of dimension 1, then X is topologically isomorphic to one of the groups $\mathbb{R}$, $\Sigma_{\boldsymbol{a}}$, and $\mathbb{T}$.

The description of distributions μ_j which are characterized by the independence of the linear forms L_1 and L_2 in the case when $X = \mathbb{R}$ is given by the Skitovich–Darmois theorem. Since the group $\Sigma_{\boldsymbol{a}}$ contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, for the group $X = \Sigma_{\boldsymbol{a}}$ this description is given by Theorem 10.3. In both cases the corresponding distributions are Gaussian. Since $\mathrm{Aut}(\mathbb{T}) = \{\pm I\}$, for the group $X = \mathbb{T}$ the description of possible distributions μ_j follows from Corollary 8.6.

Suppose that a group X is of dimension 2. If the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$, then distributions μ_j which are characterized by the independence of the linear forms L_1 and L_2 are Gaussian (Theorem 10.3). Let X be a connected locally compact Abelian group of dimension 2 and assume that the group X contains a subgroup K topologically isomorphic to the circle group $\mathbb{T}$. Then by Theorem 1.17.1 the subgroup K is a topological direct factor of X, i.e., $X = G \times K$, where G is a connected locally compact Abelian group of dimension one. Hence three cases are possible: $X \cong \mathbb{T}^2$, $X \cong \mathbb{R} \times \mathbb{T}$, and $X \cong \Sigma_{\boldsymbol{a}} \times \mathbb{T}$. The description of possible distributions μ_j for the two-dimensional torus $X = \mathbb{T}^2$ is contained in Theorem 11.5 and for the groups $X = \mathbb{R} \times \mathbb{T}$ and $X = \Sigma_{\boldsymbol{a}} \times \mathbb{T}$ is contained in Theorems 12.4 and 12.9 correspondingly. Thus we have the description of distributions μ_j of independent random variables ξ_j taking values in connected groups of dimensions 1 and 2 and having non-vanishing characteristic functions which are characterized by independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$.

Chapter V
The Skitovich–Darmois theorem for locally compact Abelian groups (the general case)

Let X be a second countable locally compact Abelian group, ξ_j, $j = 1, 2, \dots, n, n \geq 2$, be independent random variables with values in X and distributions μ_j. Consider the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$, where the coefficients α_j, β_j are topological automorphisms of X. In this chapter we continue to study some group analogues of the Skitovich–Darmois theorem. In contrast with Chapter IV we do not assume that the characteristic functions $\hat{\mu}_j(y)$ do not vanish. Let a number n of independent random variables be fixed. We focus our attention on the following problem: for which groups X does the independence of the linear forms L_1 and L_2 imply that all distributions $\mu_j \in \Gamma(X) * I(X)$? We solve this problem for different classes of groups: finite, discrete, compact totally disconnected, compact connected. We consider the case of two random variables, three random variables, and $n \geq 4$ random variables. It turns out that these cases differ from each other.

13 The number of random variables $n = 2$

Let X be a second countable locally compact Abelian group, Y be its character group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where the coefficients $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. In this section we prove that if X is a discrete group, then the independence of the linear forms L_1 and L_2 implies that $\mu_1, \mu_2 \in I(X)$. We describe compact totally disconnected groups X for which the independence of L_1 and L_2 implies that $\mu_1, \mu_2 \in I(X)$. We also prove that for an arbitrary compact connected group X there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ and independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the linear forms L_1 and L_2 are independent.

First we study the case when X is a finite group.

Theorem 13.1. *Let X be a finite group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be automorphisms of X. If the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, then $\mu_1, \mu_2 \in I(X)$.*

Proof. We note that if μ is the distribution of a random variable ξ with values in a group X and $\alpha \in \mathrm{Aut}(X)$, then by Proposition 2.10 the characteristic function of the random variable $\alpha\xi$ is equal to $\hat{\mu}(\tilde{\alpha}y)$. Taking into account the definition of the idempotent distribution (see 2.14) it follows from this that $\mu \in I(X)$ if an only if $\alpha(\mu) \in I(X)$. Therefore putting $\zeta_j = \alpha_j \xi_j$ we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, $\delta_j \in \mathrm{Aut}(X)$. We also note that the linear forms L_1 and L_2 are independent if and only if the linear forms L_1 and αL_2 are independent.

Thus in proving the theorem we may assume that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$. Put $\varepsilon = \tilde{\delta}$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.23). Set $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$ and rewrite equation (10.23). We get

$$f(u+v)g(u+\varepsilon v) = f(u)g(u)f(v)g(\varepsilon v), \quad u, v \in Y. \tag{13.1}$$

Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. It is obvious that the characteristic functions $\hat{\nu}_j(y)$ satisfy equation (10.23). Hence they also satisfy equation (13.1). If we prove that $\nu_j \in I(X)$, then 2.7 (b) and 2.7 (e) imply that $\mu_j \in I(X)$. Thus we can solve equation (13.1) assuming that $f(y) \geq 0$, $g(y) \geq 0$, $f(-y) = f(y)$, $g(-y) = g(y)$. We will prove that in this case $f(y) = g(y) = \widehat{m}_K(y)$, where K is a subgroup of X. The statement of the theorem follows from this. Set $\beta = I - \varepsilon$. Two cases are possible: $\mathrm{Ker}\,\beta = \{0\}$, $\mathrm{Ker}\,\beta \neq \{0\}$.

1. $\mathrm{Ker}\,\beta = \{0\}$. Since Y is a finite group, we have $\beta \in \mathrm{Aut}(Y)$. Substituting $v = -u$ into (13.1) we get

$$g(\beta u) = f^2(u)g(u)g(\varepsilon u), \quad u \in Y. \tag{13.2}$$

In view of $f(y) \leq 1$ and $g(y) \leq 1$ it follows from equation (13.2) that

$$g(\beta u) \leq g(u), \quad u \in Y. \tag{13.3}$$

The automorphism group $\mathrm{Aut}(Y)$ is finite because Y is a finite group. Hence

$$\beta^n = I \tag{13.4}$$

for some natural n. We assume that n in (13.4) is the smallest one. We deduce from (13.3) that

$$g(u) = g(\beta^n u) \leq \cdots \leq g(\beta u) \leq g(u), \quad u \in Y.$$

Therefore,

$$g(u) = g(\beta u) = \cdots = g(\beta^{n-1}u), \quad u \in Y.$$

Thus the function $g(y)$ takes a constant value on each of the orbits

$$O_u = \{u, \beta u, \ldots, \beta^{n-1}u\}.$$

This value generally depends on u.

Substitute $u = -\varepsilon v$ in (13.1). We get

$$f(\beta v) = f(\varepsilon v)g^2(\varepsilon v)f(v), \quad v \in Y. \tag{13.5}$$

Arguing as above we find from (13.5) that the function $f(y)$ also takes a constant value on each of the orbits O_u.

It is obvious that the group Y can be represented as a union of disjoint orbits. Denote by N the union of orbits where $g(y) > 0$, and represent Y as the union $Y = N \cup N'$, where $N' = \{y \in Y : g(y) = 0\}$. It follows from (13.2) that

$$1 = f^2(u)g(\varepsilon u), \quad u \in N. \tag{13.6}$$

This implies that $g(\varepsilon y) = 1$ for all $y \in N$. It means in particular that $\varepsilon(N) \subset N$. Since ε is a one-to-one mapping and N is a finite set, we have $\varepsilon(N) = N$. Hence $g(y) = 1$ for all $y \in N$. Taking into account that $g(y) = 0$ for all $y \in N'$, we get

$$g(y) = \begin{cases} 1 & \text{if } y \in N, \\ 0 & \text{if } y \in N'. \end{cases} \tag{13.7}$$

By Proposition 2.13, N is a subgroup of Y. Set $K = A(X, N)$. By Theorem 1.9.1, $N = A(Y, K)$. In view of 2.16 (i) it follows from (13.7) that $g(y) = \widehat{m}_K(y)$, and by 2.7 (b) that $\mu_2 = m_K$.

It also follows from (13.6) that $f(y) = 1$ for all $y \in N$. We will verify that if $y \in N'$, then $f(y) = 0$. Assume that $v \in N'$. It follows from $\varepsilon(N') = N'$ that $\varepsilon v \in N'$, and hence $g(\varepsilon v) = 0$. Then (13.5) implies that $f(\beta v) = 0$. Since $f(v) = f(\beta v)$, we have $f(v) = 0$. Thus we have proved that $f(y) = \widehat{m}_K(y)$. In case 1 the theorem is proved.

2. $\operatorname{Ker} \beta \neq \{0\}$, i.e., there exists an element $y_0 \in Y$, $y_0 \neq 0$ such that $\varepsilon y_0 = y_0$. Put $L = \operatorname{Ker} \beta = \{y \in Y : \varepsilon y = y\}$. Obviously, that $\varepsilon(L) = L$.

Consider the restriction of equation (13.1) to L. We get

$$f(u + v)g(u + v) = f(u)g(u)f(v)g(v), \quad u, v \in L. \tag{13.8}$$

Substituting $v = -u$ into (13.8) we find

$$1 = f^2(u)g^2(u), \quad u \in L.$$

It follows from this that $f(y) = g(y) = 1$ for all $y \in L$. Put $K = A(X, L)$. By Proposition 2.13 the functions $f(y)$ and $g(y)$ are L-invariant and the inclusions $\sigma(\mu_j) \subset K$, $j = 1, 2$, hold. We note that by Theorems 1.9.1 and 1.9.2, $K^* \cong Y/L$. Since the functions $f(y)$ and $g(y)$ are L-invariant, they induce some functions $\tilde{f}([y])$ and $\tilde{g}([y])$ on the factor group Y/L, namely $\tilde{f}([y]) = f(y)$, $\tilde{g}([y]) = g(y)$, $y \in [y]$. In view of $\varepsilon(L) = L$ the automorphism ε also induces some automorphism $\hat{\varepsilon}$ on the factor group Y/L by the formula $\hat{\varepsilon}[y] = [\varepsilon y]$, $y \in [y]$. Therefore we can consider equation (13.1) on the factor group Y/L. The induced homomorphism $\hat{\beta}$ can also satisfy the condition $\operatorname{Ker} \hat{\beta} \neq \{0\}$. Repeating this procedure in a finite number of steps we get the induced homomorphism $\hat{\beta}$ which is an automorphism. Then case 1 yields that $\tilde{f}([y])$ and $\tilde{g}([y])$ are the characteristic functions of some Haar distributions $m_{\widehat{K}}$, where $\widehat{K}$ is a subgroup of X. Returning to the original characteristic functions $f(y)$ and $g(y)$, we obtain the required statement. It should be noted that if $\beta^m y = 0$ for some m and for all $y \in Y$, then it means that μ_1 and μ_2 are degenerate distributions. $\square$

The proof of Theorem 13.1 implies directly the following

Corollary 13.2. *Let X be a finite group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let $\delta \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent, then $\mu_j = m_K * E_{x_j}$, where K is a subgroup of the group X and $x_j \in X$, $j = 1, 2$. Moreover $\tilde{\delta}(A(Y, K)) = A(Y, K)$.*

Using Theorem 13.1 we can prove the following statement.

Theorem 13.3. *Let $X = \mathbb{R}^m \times G$, where $m \geq 1$ and G is a finite group. Assume that ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. If the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, then $\mu_1, \mu_2 \in \Gamma(X) * I(X)$.*

Proof. We will only consider the case $m = 1$, i.e., $X = \mathbb{R} \times G$. The case when $m > 1$ can be considered similarly. We have $Y \cong \mathbb{R} \times H$, where $H = G^*$. To avoid introducing new notation we will assume that $Y = \mathbb{R} \times H$. Denote by $y = (s, h)$, $s \in \mathbb{R}$, $h \in H$, elements of the group Y. Set $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$. The subgroups $\mathbb{R}$ and H are characteristic subgroups of the group Y because $\mathbb{R}$ is the connected component of zero of Y and H consists of all compact elements of Y. Therefore every automorphism $\alpha \in \mathrm{Aut}(Y)$ is of the form $\alpha(s, h) = (\alpha s, \alpha h)$.

Arguing as in the proof of Theorem 13.1, we reduce the proof of the theorem to solving equation (13.1) which in our notation takes the form

$$f(s + s', h + h')g(s + \varepsilon s', h + \varepsilon h') = f(s, h)g(s, h)f(s', h')g(\varepsilon s', \varepsilon h'), \quad (13.9)$$

$(s, h), (s', h') \in Y$. Putting $h = h' = 0$ in (13.9) and taking into account Lemma 10.1 we find by the Skitovich–Darmois theorem

$$f(s, 0) = \exp\{-\sigma_1 s^2 + it_1 s\}, \quad g(s, 0) = \exp\{-\sigma_2 s^2 + it_2 s\}, \quad (13.10)$$

where $\sigma_j \geq 0$, $t_j \in \mathbb{R}$, $j = 1, 2$.

Putting $s = s' = 0$ in (13.9) we obtain the functional equation

$$f(0, h + h')g(0, h + \varepsilon h') = f(0, h)g(0, h)f(0, h')g(0, \varepsilon h'), \quad h,\ h' \in H. \quad (13.11)$$

We conclude from Lemma 10.1 and Corollary 13.2 that the solutions of equation (13.11) are of the form

$$f(0, h) = \widehat{m}_K(h)(g_1, h), \quad g(0, h) = \widehat{m}_K(h)(g_2, h), \quad h \in H, \quad (13.12)$$

where K is a subgroup of the group G and $g_1, g_2 \in G$. Set $E = A(H, K)$. Consider the shifts $\mu_j' = \mu_j * E_{-g_j}$ and put $f'(y) = \hat{\mu}_1'(y)$, $g'(y) = \hat{\mu}_2'(y)$. In view of 2.14 (i) and 2.7 (c) it follows from (13.12) that

$$f'(0, h) = g'(0, h) = \begin{cases} 1 & \text{if } h \in E, \\ 0 & \text{if } h \notin E. \end{cases} \quad (13.13)$$

Obviously, the characteristic functions $f'(y)$ and $g'(y)$ satisfy equation (13.9). Put $B = H/E$. We have $Y/E \cong \mathbb{R} \times B$. By Theorem 1.9.2, $K^* \cong B$, and hence $(\mathbb{R} \times K)^* \cong Y/E$. By Proposition 2.13 we deduce from (13.13) that the characteristic functions $f'(y)$ and $g'(y)$ are E-invariant. By Corollary 13.2 we have $\varepsilon(E) = E$. Hence we can pass from equation (13.9) for the functions $f'(y)$ and $g'(y)$ on the group Y to the induced equation on the factor group Y/E putting $\tilde{f}([y]) = f'(y)$, $\tilde{g}([y]) = g'(y)$, $\hat{\varepsilon}[y] = [\varepsilon y]$. It follows from $\varepsilon(E) = E$ that $\hat{\varepsilon} \in \mathrm{Aut}(Y/E)$. It means that we pass from consideration of the random variables ξ_j with values in the group $\mathbb{R} \times G$ to consideration of random variables taking values in the group $\mathbb{R} \times K$. Obviously, we have

$$\{b \in B : \tilde{f}(0, b) = 1\} = \{b \in B : \tilde{g}(0, b) = 1\} = \{0\}. \tag{13.14}$$

Put $A = \{b \in B : \hat{\varepsilon} b = b\}$. As has been proved in case 2 of Theorem 13.1 it follows from (13.11) that $\tilde{f}(0, b) = \tilde{g}(0, b) = 1$ for all $b \in A$. In view of (13.14) it means that $A = \{0\}$. This implies that

$$I - \hat{\varepsilon} \in \mathrm{Aut}(Y/E). \tag{13.15}$$

Moreover,

$$\tilde{f}(0, b) = \tilde{g}(0, b) = \begin{cases} 1 & \text{if } b = 0, \\ 0 & \text{if } b \neq 0. \end{cases} \tag{13.16}$$

Consider equation (13.9) for the functions $\tilde{f}(s, b)$ and $\tilde{g}(s, b)$ on the factor group $Y/E \cong \mathbb{R} \times B$.

$$\begin{aligned} &\tilde{f}(s + s', b + b')\tilde{g}(s + \hat{\varepsilon}s', b + \hat{\varepsilon}b') \\ &\quad = \tilde{f}(s, b)\tilde{g}(s, b)\tilde{f}(s', b')\tilde{g}(\hat{\varepsilon}s', \hat{\varepsilon}b'), \quad (s, b), (s', b') \in \mathbb{R} \times B. \end{aligned} \tag{13.17}$$

We note that

$$\tilde{f}(s, 0) = f(s, 0), \quad \tilde{g}(s, 0) = g(s, 0). \tag{13.18}$$

Putting $s = 0,\ b' = 0$ in (13.17) we get

$$\tilde{f}(s', b)\tilde{g}(\hat{\varepsilon}s', b) = \tilde{f}(0, b)\tilde{g}(0, b)\tilde{f}(s', 0)\tilde{g}(\hat{\varepsilon}s', 0), \quad s' \in \mathbb{R},\ b' \in B. \tag{13.19}$$

We conclude from (13.16) that the right-hand side of equation (13.19) is equal to zero for all $b \neq 0$. Hence $\tilde{f}(s', b)\tilde{g}(\hat{\varepsilon}s', b) = 0$ for all $s' \in \mathbb{R}$. By Proposition 2.20 it follows from (13.10) and (13.18) that $\tilde{f}(s, b)$ and $\tilde{g}(s, b)$ are entire functions in s for every fixed $b \in B$. Therefore either $\tilde{f}(s, b) \equiv 0$ or $\tilde{g}(s, b) \equiv 0$ in $s \in \mathbb{R}$ for all $b \neq 0$. Hence the right-hand side of equation (13.17) is equal to zero for all $b \neq 0$. Take an arbitrary element $b_0 \in B$, $b_0 \neq 0$ and find b and b' from the system of equations

$$\begin{cases} b + b' = 0, \\ b + \varepsilon b' = b_0. \end{cases}$$

In view of (13.15) this system of equations has a unique solution. Substituting the found b, b', and $s' = 0$ into (13.17) and taking into account (13.10), we obtain that $\tilde{g}(s, b_0) = 0$ for all $s \in \mathbb{R}$. By (13.10) and (13.18) it follows from this that

$$\tilde{g}(s,b) = \begin{cases} \exp\{-\sigma_2 s^2 + it_2 s\} & \text{if } b = 0, \\ 0 & \text{if } b \neq 0. \end{cases}$$

Returning to the original function $g(s, h)$, we find

$$g(s,h) = \begin{cases} \exp\{-\sigma_2 s^2 + it_2 s\}(g_2, h) & \text{if } h \in E, \\ 0 & \text{if } h \notin E. \end{cases} \tag{13.20}$$

We also obtain a similar representation for the function $f(s, h)$. Let γ_j be a Gaussian distribution on the group X with the characteristic function

$$\hat{\gamma}_j(y) = \exp\{-\sigma_j s^2 + it_j s\}(g_j, h), \quad (s,h) \in Y, \ j = 1, 2.$$

We deduce from (13.20) and 2.14 (i) that $g(s, h) = \hat{\gamma}_2(s, h)\widehat{m}_K(s, h)$. Then 2.7 (b) implies that $\mu_2 = \gamma_2 * m_K$. Arguing as above we also prove that $\mu_1 = \gamma_1 * m_K$. □

Corollary 13.4. *Let $X = \mathbb{R}^m \times G$, where $m \geq 1$ and G is a finite group. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let $\delta \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent, then $\mu_j = \gamma_j * m_F * E_{x_j}$, where $\gamma_j \in \Gamma^s(\mathbb{R}^m)$, F is a subgroup of the group G, and $x_j \in X$, $j = 1, 2$.*

Now consider the case when X is a discrete group. First we prove a statement which can be regarded as a group analogue of the Skitovich–Darmois theorem for discrete torsion-free groups.

Theorem 13.5. *Let X be a discrete torsion-free group, ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent, then all μ_j are degenerate distributions.*

Proof. Taking into consideration new independent random variables $\zeta_j = \alpha_j\xi_j$ we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \dots + \xi_n$ and $L_2 = \delta_1\xi_1 + \dots + \delta_n\xi_n$, where $\delta_j \in \mathrm{Aut}(X)$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.2).

First we prove that the characteristic functions $\hat{\mu}_j(y)$ do not vanish. Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.2). Set

$$A_j = \{y \in Y : \hat{\nu}_j(y) > 0\}, \qquad A = \bigcap_{j=1}^{n} A_j,$$

$$B = \Big\{y \in Y : \prod_{j=1}^{n} \hat{\nu}_j(\tilde{\delta}_j y) > 0\Big\}, \quad C = \bigcap_{j=1}^{n} \tilde{\delta}_j(B).$$

It is easy to see that $A + C \subset A$. Indeed, let $u \in A$, $v \in C$. Then $v = \tilde{\delta}_j t_j$, where $t_j \in B$, $j = 1, 2, \dots, n$. It results from equation (10.2) that $u + v = u + \tilde{\delta}_j t_j \in A_j$, $j = 1, 2, \dots, n$. Therefore $u + v \in \bigcap_{j=1}^n A_j = A$. Put $(m)C = \{y \in Y : y = y_1 + \dots + y_m,\ y_j \in C\}$. Then the inclusion $A + C \subset A$ implies that $A + \bigcup_{m=1}^\infty (m)C \subset A$. Hence $A + \bigcup_{m=1}^\infty (m)C = A$ because $0 \in C$.

Since X is a discrete torsion-free group, by Theorems 1.6.1 and 1.6.2, Y is a compact connected group. It follows from the connectedness of the group Y that $Y = \bigcup_{m=1}^\infty (m)C$ because C is an open set containing the zero of the group Y. This implies that $A = Y$, i.e., the characteristic functions $\hat{\nu}_j(y) > 0$ for all $y \in Y$. Hence $\hat{\mu}_j(y)$, $j = 1, 2, \dots, n$, do not vanish.

Put $\varphi_j(y) = -\ln \widehat{\nu_j(y)}$. It results from (10.2) that the functions $\varphi_j(y)$ satisfy the equation

$$\sum_{j=1}^n \varphi_j(u + \tilde{\delta}_j v) = \sum_{j=1}^n \varphi_j(u) + \sum_{j=1}^n \varphi_j(\tilde{\delta}_j v), \quad u, v \in Y. \tag{13.21}$$

Integrating equation (13.21) over the group Y with respect to the Haar distribution $dm_Y(u)$ and using that the Haar distribution m_Y is Y-invariant, we find

$$\sum_{j=1}^n \varphi_j(\tilde{\delta}_j v) = 0, \quad v \in Y. \tag{13.22}$$

It follows from this that all functions $\varphi_j(y) \equiv 0$ on Y. This implies that the characteristic functions $\hat{\nu}_j(y) \equiv 1$ on Y, $j = 1, 2, \dots, n$. Thus we have proved that all $\nu_j = E_0$. Hence $\mu_j \in D(X)$, $j = 1, 2, \dots, n$. □

Corollary 13.6. *Let Y be a compact connected group and $\tilde{\alpha}_j$, $\tilde{\beta}_j \in \mathrm{Aut}(Y)$, $j = 1, 2, \dots, n$, $n \geq 2$. Let $\hat{\mu}_j(y)$ be characteristic functions on the group Y satisfying equation 10.1* (i). *Then $\hat{\mu}_j(y)$ are of the form*

$$\hat{\mu}_j(y) = (x_j, y), \quad x_j \in X,\ j = 1, 2, \dots, n.$$

Remark 13.7. We proved in Theorem 13.5 that the characteristic functions $\hat{\mu}_j(y)$ do not vanish. In the course of the proof we used only the connectedness of the group Y. This implies in particular that if $Y = \mathbb{R}^m$ and characteristic functions $\hat{\mu}_j(y)$ on the group Y satisfy equation 10.1 (i), then $\hat{\mu}_j(y)$ do not vanish. Taking into account this remark we see that the Ghurye–Olkin theorem ([54]) results from Lemma 10.1 and Theorem 10.3.

Theorem 13.5 allows us to prove the following general statement.

Proposition 13.8. *Let $X = \mathbb{R}^m \times G$, where $m \geq 0$, and the group G contains a compact open subgroup. Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. Let α_j, $\beta_j \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent. Then the random variables ξ_j can be replaced by their shifts ξ'_j in such a way that all distributions μ'_j are supported in $\mathbb{R}^m \times K$, where K is a compact subgroup of the group X.*

Proof. Obviously, we can assume without loss of generality that $L_1 = \xi_1+\cdots+\xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, where $\delta_j \in \mathrm{Aut}(X)$. Note that $b_X = b_G$. By Theorem 1.11.2 we have $c_Y = L\times M$, where $L \cong \mathbb{R}^m$, and M is a connected compact group. Since by Theorem 1.9.3, $A(G, M) = b_G$, we conclude that $A(X, M) = \mathbb{R}^m \times b_G = \mathbb{R}^m \times b_X$.

Put $\nu_j = \mu_j * \bar{\mu}_j$. By Lemma 10.1 it follows from the independence of L_1 and L_2 that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.2). We find from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. It is clear that the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.2). Note that M is a characteristic subgroup of the group Y. Hence we can consider the restriction of equation (10.2) to the subgroup M. By Corollary 13.6, $\hat{\nu}_j(y) = 1$ for $y \in M$. By Proposition 2.13, $\sigma(\nu_j) \subset A(X, M) = \mathbb{R}^m \times b_X$. It follows from Proposition 2.2 that the distributions μ_j can be replaced by their shifts μ'_j in such a manner that $\sigma(\mu'_j) \subset \mathbb{R}^m \times b_X$. It should be noted that $\mathbb{R}^m \times b_X$ is a characteristic subgroup and by Corollary 10.2 the linear forms $L'_1 = \alpha_1\xi'_1 + \cdots + \alpha_n\xi'_n$ and $L'_2 = \beta_1\xi'_1 + \cdots + \beta_n\xi'_n$ are independent.

Thus we may prove the proposition assuming that G consists of compact elements. Then by Theorem 1.9.3, the group $H = G^*$ is totally disconnected. By Theorem 1.12.1 every neighbourhood of zero of the group H contains a compact open subgroup. Denote by N this subgroup and choose it in such a way that all characteristic functions $\hat{\nu}_j(y) > 0$ for $y \in N$. Applying Theorem 1.12.1 again and taking into account the continuity of automorphisms $\tilde{\delta}_j$ we get that there exists a compact open subgroup $F \subset N$ such that $\tilde{\delta}_j(F) \subset N$, $j = 1, 2, \ldots, n$. Set $\varphi_j(y) = -\ln \hat{\nu}_j(y)$, $y \in H$. Since the characteristic functions $\hat{\nu}_j(y)$ satisfy equation (10.2), the functions $\varphi_j(y)$ satisfy equation (13.21) for $u \in N$, $v \in F$.

Integrating equation (13.21) over the group N with respect to the Haar distribution $dm_N(u)$ and using that the Haar distribution m_N is N-invariant, we find that the functions $\varphi_j(y)$ satisfy equation (13.22) on the subgroup F. Hence all characteristic functions $\hat{\nu}_j(\tilde{\delta}_j v) \equiv 1$ on F. Put $B = \bigcap_{j=1}^{s} \widetilde{\delta_j(F)}$. Then B is an open subgroup of H, and all characteristic functions $\hat{\nu}_j(y) \equiv 1$ on B.

Put $K = A(G, B)$ and note that $A(X, B) = \mathbb{R}^m \times K$, where K is a compact group by Theorem 1.9.4. By Proposition 2.13, $\sigma(\nu_j) \subset \mathbb{R}^m \times K$. It follows from Proposition 2.2 that the distributions μ_j can be replaced by their shifts μ'_j in such a manner that all $\sigma(\mu'_j) \subset \mathbb{R}^m \times K$. It should be noted that generally speaking the subgroup $\mathbb{R}^m \times K$ does not need to be characteristic. □

The following statement follows directly from the proof of Proposition 13.8.

Lemma 13.9. *Let X be a discrete torsion group, ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j such that all characteristic functions $\hat{\mu}_j(y) \geq 0$. Let $\delta_j \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$ are independent, then all characteristic functions $\hat{\mu}_j(y) \equiv 1$ on an open subgroup $B \subset Y$.*

Lemma 13.10. *Let K be a compact subgroup of a group X and α be a continuous endomorphism of the group X. Then the following statements are equivalent:*

(i) *if* $\tilde{\alpha} y \in A(Y, K)$, *then* $y \in A(Y, K)$;
(ii) $\alpha(K) \supset K$.

Proof. Let us assume that (i) is true. Suppose that $y \in A(Y, \alpha(K))$. Then $(\alpha x, y) = 1$ for all $x \in K$. Hence $(x, \tilde{\alpha} y) = 1$ for all $x \in K$, i.e., $\tilde{\alpha} y \in A(Y, K)$, and in view of (i), $y \in A(Y, K)$. Therefore $A(Y, \alpha(K)) \subset A(Y, K)$. By Theorem 1.9.1 it follows from this that (ii) holds.

Let us assume that (ii) is true. We conclude from (ii) that

$$A(Y, K) \supset A(Y, \alpha(K)). \tag{13.23}$$

Suppose that $\tilde{\alpha} y \in A(Y, K)$. Then $(x, \tilde{\alpha} y) = 1$ for all $x \in K$. By 1.13 (a), $(\alpha x, y) = 1$ for all $x \in K$, i.e., $y \in A(Y, \alpha(K))$. Hence (13.23) implies that $y \in A(Y, K)$. The equivalence of (i) and (ii) is proved. □

Lemma 13.11. *Let G be a closed subgroup of a group X and $\delta \in \mathrm{Aut}(X)$. Then the following statements are equivalent:*

(i) $\delta(G) = G$;
(ii) $\tilde{\delta}(A(Y, G)) = A(Y, G)$.

Proof. In view of Theorem 1.9.1 and 1.13 (a) it suffices to show that (i) implies (ii). Let us assume that (i) is true and let $y \in A(Y, G)$. Then $(\delta x, y) = 1$ for all $x \in G$ and hence $(x, \tilde{\delta} y) = 1$ for all $x \in G$, i.e., $\tilde{\delta} y \in A(Y, G)$. Thus

$$\tilde{\delta}(A(Y, G)) \subset A(Y, G). \tag{13.24}$$

We note that the equality $\delta^{-1}(G) = G$ follows from (i). As has been shown above $\tilde{\delta}^{-1}(A(Y, G)) \subset A(Y, G)$. It follows from this that $A(Y, G) \subset \tilde{\delta}(A(Y, G))$. Taking into account (13.24), we get (ii). □

Corollary 13.12. *A closed subgroup G of a group X is characteristic if and only if its annihilator $A(Y, G)$ is a characteristic subgroup of the group Y.*

Lemma 13.13. *Let ξ_1 and ξ_2 be independent identically distributed random variables with values in a group X and distribution m_K, where K is a compact subgroup of X. Let $\delta \in \mathrm{Aut}(X)$. Then the following statements are equivalent:*

(i) *the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent;*
(ii) $(I - \delta)(K) \supset K$.

Proof. Put $\varepsilon = \tilde{\delta}$, $\beta = I - \varepsilon$, $H = A(Y, K)$, $f(y) = \widehat{m}_K(y)$. Let us assume that (i) is true. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$f(u + v) f(u + \varepsilon v) = f^2(u) f(v) f(\varepsilon v), \quad u, v \in Y. \tag{13.25}$$

Substituting $v = -u$ into equation (13.25) we obtain

$$f(\beta u) = f^3(u) f(\varepsilon u), \quad u \in Y. \tag{13.26}$$

It follows from (13.26) that if $\beta y \in H$, then $y \in H$. Therefore by Lemma 13.10, (ii) holds.

Let us assume that (ii) is true. We will verify that the function $f(y) = \widehat{m}_K(y)$ satisfies equation (13.25). If $u, v \in H$ and $\varepsilon v \in H$, then both sides of equation (13.25) are equal to 1. If $u, v \in H$ and $\varepsilon v \notin H$, then both sides of equation (13.25) are equal to zero. If either $u \in H$, $v \notin H$ or $u \notin H$, $v \in H$, then both sides of equation (13.25) are equal to zero. If $u, v \notin H$, then the right-hand side of equation (13.25) is equal to zero. If the left-hand side of equation (13.25) is not equal to zero we have $u + v, u + \varepsilon v \in H$. It follows from this that $\beta v \in H$. But in view of Lemma 13.10, (ii) implies that assertion 13.10 (i) holds for $\tilde{\alpha} = \beta$. Hence $v \in H$. The contradiction obtained shows that the left-hand side of equation (13.25) is also equal to zero. Thus the function $f(y)$ satisfies equation (13.25). By Lemma 10.1 the linear forms L_1 and L_2 are independent. The equivalence of (i) and (ii) is proved. □

Remark 13.14. We note that when $\delta = -I$, Lemma 13.13 yields the description of idempotent distributions m_K which are Gaussian distributions in the sense of Bernstein on the group X. Namely, K must be a Corwin group (see Proposition 7.4).

We also note that if K is a finite group, then (ii) is equivalent to $(I - \delta)(K) = K$. On the other hand, if K is a compact group, then in general, the independence of the linear forms L_1 and L_2 does not imply the equality $(I - \delta)(K) = K$. Indeed, let G be an arbitrary compact group. Consider the direct product

$$X = \mathop{\mathbf{P}}_{j=-\infty}^{\infty} G_j,$$

where all $G_j \cong G$. Put

$$K = \mathop{\mathbf{P}}_{j=0}^{\infty} G_j,$$

and let $\delta \in \mathrm{Aut}(X)$ be an automorphism of the form

$$\delta(g_j)_{j=-\infty}^{\infty} = (g_{j+1})_{j=-\infty}^{\infty}, \quad (g_j)_{j=-\infty}^{\infty} \in X.$$

It is obvious that (ii) is true, whereas K is a proper subgroup of $(I - \delta)(K)$.

Lemma 13.15. *Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions $\mu_1 = m_{K_1}$ and $\mu_2 = m_{K_2}$, where K_1 and K_2 are finite subgroups of X. Let $\delta \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent, then $K_1 = K_2 = K$ and $\delta(K) = K$.*

Proof. Put $\varepsilon = \tilde{\delta}$, $\beta = I - \varepsilon$, $f(y) = \widehat{m}_{K_1}(y)$, $g(y) = m_{K_2}(y)$, $H_j = A(Y, K_j)$, $j = 1, 2$. We use representation 2.14 (i). By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $f(y)$ and $g(y)$ satisfy equation 10.1 (i) which take the form (13.1). Substituting $u = -\varepsilon v$ into (13.1) we get (13.5). We conclude from equation (13.5) that if $\beta v \in H_1$, then $v \in H_1$. Hence by Lemma 13.10, $(I - \delta)(K_1) \supset K_1$. Taking into account that subgroup K_1 is finite, we get $(I - \delta)(K_1) = K_1$. This implies that $\delta(K_1) \subset K_1$. Since $\delta \in \mathrm{Aut}(X)$ and K_1 is a finite group, we have $\delta(K_1) = K_1$.

By Lemma 13.11 it follows from this that $\varepsilon(H_1) = H_1$. Consider the restriction of equation (13.1) to the subgroup H_1. We have

$$g(u + \varepsilon v) = g(u)g(\varepsilon v), \quad u, v \in H_1.$$

This implies that $g(y) = 1$ for all $y \in H_1$, i.e., $H_1 \subset H_2$.

Substituting $v = -u$ into equation (13.1) we obtain (13.2). Arguing as above and considering equation (13.2) instead of equation (13.5), we prove that $H_2 \subset H_1$. Hence $H_1 = H_2$. It follows from this by Theorem 1.9.1 that $K_1 = K_2 = K$. □

Remark 13.16. In general Lemma 13.15 is false if K_1 and K_2 are compact but not finite groups. To show this consider the following example. Let X be the same group as in Remark 13.14. Put $H = G^*$. By Theorem 1.7.2,

$$Y = \mathop{\mathbf{P}^*}_{j=-\infty}^{\infty} H_j,$$

where $H_j \cong H$. Let $\delta \in \mathrm{Aut}(X)$ be an automorphism of the form

$$\delta(g_j)_{j=-\infty}^{\infty} = (g_{j-2})_{j=-\infty}^{\infty}, \quad (g_j)_{j=-\infty}^{\infty} \in X.$$

Put $\varepsilon = \tilde{\delta}$. Then

$$\varepsilon(h_j)_{j=-\infty}^{\infty} = (h_{j+2})_{j=-\infty}^{\infty}, \quad (h_j)_{j=-\infty}^{\infty} \in Y.$$

It is easy to see that

$$\mathrm{Ker}(I - \varepsilon) = \{0\}. \tag{13.27}$$

Consider the subgroups

$$K_1 = \mathop{\mathbf{P}}_{j \neq 1} G_j, \quad K_2 = \mathop{\mathbf{P}}_{j \neq 2} G_j.$$

It is obvious that $K_1 \neq K_2$ and $H_j = A(Y, K_j)$, $j = 1, 2$. We will check that the characteristic functions $f(y) = \widehat{m}_{K_1}(y)$ and $g(y) = \widehat{m}_{K_2}(y)$ satisfy equation (13.1). It suffices to verify that equation (13.1) is satisfied for all $u \neq 0$, $v \neq 0$. Since $H_1 \cap H_2 = \{0\}$, the right-hand side of equation (13.1) is equal to zero for all $u \neq 0$, $v \neq 0$. Then the left-hand side of equation (13.1) is also equal to zero. In the opposite case we have $u + v \in H_1$, $u + \varepsilon v \in H_2$. It follows from this that $(I - \varepsilon)v \in H_1 \times H_2$. But it is possible only if $(I - \varepsilon)v = 0$. In view of (13.27) this implies that $v = 0$, contrary to the assumption. Thus the characteristic functions $f(y)$ and $g(y)$ satisfy equation (13.1). By Lemma 10.1 if ξ_1 and ξ_2 are independent random variables with values in X and distributions m_{K_1} and m_{K_2}, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent.

Theorem 13.17. *Let X be a discrete group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be automorphisms of X. If the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, then $\mu_1, \mu_2 \in I(X)$.*

Proof. Reasoning as in the proof of Theorem 13.1, we reduce the proof of the theorem to the case when $L_1 = \xi_1+\xi_2$ and $L_2 = \xi_1+\delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$. Set $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (13.1), where $\varepsilon = \tilde{\delta}$. Put $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (13.1). If we prove that $\nu_j \in I(X)$, then 2.7 (b) and 2.7 (e) imply that $\mu_j \in I(X)$. Thus we can solve equation (13.1) assuming that $f(y) \geq 0$, $g(y) \geq 0$, $f(-y) = f(y)$, $g(-y) = g(y)$. We will prove that in this case $f(y) = g(y) = \widehat{m}_K(y)$, where K is a finite subgroup of the group X. The statement of the theorem follows from this.

Since X is a discrete group, b_X is a subgroup of X consisting of elements of finite order. Taking into account that b_X is a characteristic subgroup of X and applying Proposition 13.8 we can assume from the beginning that X is a torsion group.

Put $E_f = \{y \in Y : f(y) = 1\}$, $E_g = \{y \in Y : g(y) = 1\}$. Then by Proposition 2.13, $\sigma(\mu_1) \subset A(X, E_f) = F$, $\sigma(\mu_2) \subset A(X, E_g) = G$. In view of Lemma 13.9 there exists an open subgroup B of the group Y such that $B \subset E_f \cap E_g$. Set $S = A(X, B)$. Then F and G are subgroups of S. Since B is an open subgroup, by Theorem 1.9.4, S is a compact group. Taking into account that X is a discrete group, S is a finite group. Hence F and G are also finite groups.

Note now that for all natural n the functions $f^n(y)$ and $g^n(y)$ also satisfy equation (13.1), i.e.,

$$f^n(u+v)g^n(u+\varepsilon v) = f^n(u)g^n(u)f^n(v)g^n(\varepsilon v), \quad u, v \in Y. \tag{13.28}$$

Obviously, there exist the limits

$$\bar{f}(y) = \lim_{n\to\infty} f^n(y) = \begin{cases} 1 & \text{if } y \in E_f, \\ 0 & \text{if } y \notin E_f, \end{cases} \quad \bar{g}(y) = \lim_{n\to\infty} g^n(y) = \begin{cases} 1 & \text{if } y \in E_g, \\ 0 & \text{if } y \notin E_g. \end{cases}$$

Since by Theorem 1.9.1, $E_f = A(Y, F)$ and $E_g = A(Y, G)$, it follows from 2.14 (i) that

$$\widehat{m}_F(y) = \begin{cases} 1 & \text{if } y \in E_f, \\ 0 & \text{if } y \notin E_f, \end{cases} \quad \widehat{m}_G(y) = \begin{cases} 1 & \text{if } y \in E_g, \\ 0 & \text{if } y \notin E_g. \end{cases}$$

Hence

$$\widehat{m}_F(y) = \bar{f}(y), \quad \widehat{m}_G(y) = \bar{g}(y).$$

Let ζ_1 and ζ_2 be independent random variables with values in X and distributions $\lambda_1 = m_F$ and $\lambda_2 = m_G$. We deduce from (13.28) that the characteristic functions $\bar{f}(y)$ and $\bar{g}(y)$ also satisfy equation (13.1). By Lemma 10.1 this implies that the linear forms $L_1 = \zeta_1 + \zeta_2$ and $L_2 = \zeta_1 + \delta\zeta_2$ are independent. Note that the conditions of Lemma 13.15 are fulfilled. It follows from Lemma 13.15 that $F = G$ and $\delta(G) = G$.

Let us return to the original random variables ξ_1 and ξ_2 and to the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$. Since $\sigma(\mu_j) \subset G$, the random variables ξ_j take values in the finite group G. Since $\delta(G) = G$, the conditions of Corollary 13.2 are fulfilled. By Corollary 13.2, $\mu_j = m_K * E_{g_j}$, where K is a subgroup of the group G, and $g_j \in G$, $j = 1, 2$. □

Corollary 13.18. *Let X be a discrete group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let $\delta \in \mathrm{Aut}(X)$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent, then $\mu_j = m_K * E_{x_j}$, where K is a finite subgroup of the group X and $x_j \in X$, $j = 1, 2$. Moreover $\tilde{\delta}(A(Y, K)) = A(Y, K)$.*

Remark 13.19. Theorem 13.17 implies an analogue of Theorem 13.3 for groups of the form $X = \mathbb{R}^m \times G$, where $m \geq 1$ and G is a discrete group. The proof is the same as the proof of Theorem 13.3, but instead of Corollary 13.2 we use Corollary 13.18.

Now consider the case when X is a compact totally disconnected group. We need some lemmas.

Lemma 13.20. *Let X be a compact group. Assume that there exist an automorphism $\delta \in \mathrm{Aut}(X)$ and an element $\tilde{y} \in Y$ such that the following conditions are satisfied:*

(i) $\mathrm{Ker}(I - \tilde{\delta}) = \{0\}$;
(ii) $(I - \tilde{\delta})Y \cap \{0, \pm\tilde{y}, \pm 2\tilde{y}\} = \{0\}$;
(iii) $\tilde{\delta}\tilde{y} \neq -\tilde{y}$.

Then for every $n \geq 2$ there exist independent identically distributed random variables ξ_j, $j = 1, 2, \ldots, n$, with values in X and distribution $\mu \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-1} + \delta\xi_n$ are independent.

Proof. Consider on the group X the function

$$\rho(x) = 1 + (1/2)\,\mathrm{Re}(x, \tilde{y}).$$

Then $\rho(x) > 0$, $x \in X$, and

$$\int_X \rho(x) d m_X(x) = 1.$$

Denote by μ the distribution on X with density $\rho(x)$ with respect to the Haar distribution m_X. It is obvious that $\mu \notin \Gamma(X) * I(X)$. Let ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent identically distributed random variables ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, with values in X and distribution μ. Put $f(y) = \hat{\mu}(y)$. We will verify that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-1} + \delta\xi_n$ are independent. By Lemma 10.1 it suffices to show that the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$f^{n-1}(u + v) f(u + \varepsilon v) = f^n(u) f^{n-1}(v) f(\varepsilon v), \quad u, v \in Y, \tag{13.29}$$

where $\varepsilon = \tilde{\delta}$. Let $a(\tilde{y}) = \frac{1}{4}$ if $2\tilde{y} \neq 0$ and $a(\tilde{y}) = \frac{1}{2}$ if $2\tilde{y} = 0$. It is easily seen that

$$f(y) = \begin{cases} 1 & \text{if } y = 0, \\ a(\tilde{y}) & \text{if } y = \pm\tilde{y}, \\ 0 & \text{if } y \notin \{0, \pm\tilde{y}\}. \end{cases} \tag{13.30}$$

Obviously, it suffices to show that equation (13.29) is satisfied for all $u \neq 0$, $v \neq 0$. We conclude from (i) and (iii) that $\varepsilon\tilde{y} \neq \pm\tilde{y}$. Hence (13.30) implies that the right-hand side of equation (13.29) is equal to zero for all $v \neq 0$. If the left-hand side of equation (13.29) is not equal to zero, this implies that $u + v, u + \varepsilon v \in \{0, \pm\tilde{y}\}$. It follows from this that $(I - \varepsilon)v \in \{0, \pm\tilde{y}, \pm 2\tilde{y}\}$. Taking into account (ii), we obtain that $(I - \varepsilon)v = 0$. But then (i) implies that $v = 0$. The contradiction obtained proves that the left-hand side of equation (13.29) is also equal to zero. □

Lemma 13.21. *Let p be a prime number and $X = \Delta_p^{\aleph_0}$. Then for every $n \geq 2$ there exist an automorphism $\delta \in \mathrm{Aut}(X)$ and independent identically distributed random variables ξ_j, $j = 1, 2, \dots, n$, with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-1} + \delta\xi_n$ are independent.*

Proof. We will assume that elements of the group X are bilateral sequences of the form $x = (x_k)_{k=-\infty}^{\infty}$, $x_k \in \Delta_p$. Consider $\delta \in \mathrm{Aut}(X)$ of the form

$$\delta(x_k)_{k=-\infty}^{\infty} = (x_{k-1})_{k=-\infty}^{\infty}.$$

We have $\Delta_p^* \cong \mathbb{Z}(p^\infty)$. Then by Theorem 1.7.2, $Y \cong (\mathbb{Z}(p^\infty))^{\aleph_0 *}$. Denote by $y = (y_k)_{k=-\infty}^{\infty}$, $y_k \in \mathbb{Z}(p^\infty)$, elements of the group Y. Then

$$\tilde{\delta}(y_k)_{k=-\infty}^{\infty} = (y_{k+1})_{k=-\infty}^{\infty}.$$

Put $\tilde{y} = (y_k)_{k=-\infty}^{\infty}$, where $y_k = 0$ for all $k \neq 0$, and y_0 is an arbitrary nonzero element of the group $\mathbb{Z}(p^\infty)$. We will show that the automorphism δ and the element $\tilde{y}$ satisfy the conditions of Lemma 13.20.

Take $y = (y_k)_{k=-\infty}^{\infty} \in \mathrm{Ker}(I - \tilde{\delta})$. Then $\tilde{\delta}y = y$. It follows from this that $y_{k+1} = y_k, k \in \mathbb{Z}$. Since $y_k \neq 0$ only for a finite number of indices k, we have $y_k = 0$, $k \in \mathbb{Z}$. Thus 13.20 (i) is fulfilled. Note that if $y = (y_k)_{k=-\infty}^{\infty} \in (I - \tilde{\delta})Y$, $y \neq 0$, then at least two elements y_{k_1} and y_{k_2} are different from zero. This implies 13.20 (ii). Obviously, 13.20 (iii) is also fulfilled. Since the group X is totally disconnected, by Proposition 3.6, $\Gamma(X) = D(X)$. Hence $\Gamma(X) * I(X) = I(X)$, and the statement of the lemma follows from Lemma 13.20. □

Lemma 13.22. *Let p be a prime number and*

$$X = \mathop{\mathbf{P}}_{m=1}^{\infty} \mathbb{Z}(p^{k_m}), \quad k_m \leq k_{m+1},\ m = 1, 2, \dots.$$

Then for every $n \geq 2$ there exist an automorphism $\delta \in \mathrm{Aut}(X)$ and independent identically distributed random variables ξ_j, $j = 1, 2, \dots, n$, with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-1} + \delta\xi_n$ are independent.

Proof. Denote elements of the group X by $t = (t_m)_{m=1}^{\infty}$, where $t_m \in \mathbb{Z}(p^{k_m})$. Let $j \geq i$ and π_{ij} be the epimorphism $\pi_{ij} : \mathbb{Z}(p^{k_j}) \mapsto \mathbb{Z}(p^{k_i})$ defined by

$$\pi_{ij}(t_j) = t_j \pmod{p^{k_i}}, \quad t_j \in \mathbb{Z}(p^{k_j}).$$

Note that $\pi_{m,m+1} \circ \pi_{m+1,m+2} = \pi_{m,m+2}$.

Define a homomorphism $\delta\colon X \mapsto X$ by the formula $\delta(t_m)_{m=1}^{\infty} = (s_m)_{m=1}^{\infty}$, where

$$s_m = \begin{cases} t_m + \pi_{m,m+1}t_{m+1} + \pi_{m,m+2}t_{m+2} & \text{if } m \text{ is odd,} \\ t_m + \pi_{m,m+1}t_{m+1} & \text{if } m \text{ is even.} \end{cases} \tag{13.31}$$

It is obvious that the homomorphism δ is continuous. We will show that δ is a monomorphism. Assume that $\delta(t_m)_{m=1}^{\infty} = 0$. Take two sequential numbers m and $m+1$ with m odd. We have

$$t_m + \pi_{m,m+1}t_{m+1} + \pi_{m,m+2}t_{m+2} = 0, \tag{13.32}$$

$$t_{m+1} + \pi_{m+1,m+2}t_{m+2} = 0. \tag{13.33}$$

Apply $\pi_{m,m+1}$ to equality (13.33). We obtain

$$\pi_{m,m+1}t_{m+1} + \pi_{m,m+2}t_{m+2} = 0. \tag{13.34}$$

Subtracting (13.34) from (13.32) we get $t_m = 0$ $(m = 1, 3, 5, \dots)$. Then it follows from (13.33) that $t_{m+1} = 0$ $(m = 1, 3, 5, \dots)$. Thus all $t_m = 0$.

We will verify that δ is an epimorphism. For this purpose we will prove that for any $s = (s_m)_{m=1}^{\infty} \in X$ the equation $\delta t = s$ has a solution. It suffices to show the existence of the solution of the system of equations

$$\begin{cases} t_m + \pi_{m,m+1}t_{m+1} + \pi_{m,m+2}t_{m+2} = s_m, & (13.35) \\ t_{m+1} + \pi_{m+1,m+2}t_{m+2} = s_{m+1} & (13.36) \end{cases}$$

for any odd m. Apply $\pi_{m,m+1}$ to equality (13.36). We obtain

$$\pi_{m,m+1}t_{m+1} + \pi_{m,m+2}t_{m+2} = \pi_{m,m+1}s_{m+1}. \tag{13.37}$$

Subtracting (13.37) from (13.35) we get that $t_m = s_m - \pi_{m,m+1}s_{m+1}$ and find in such a way all t_m, where m is odd, and then we find from (13.36) all t_m, where m is even. Thus we have proved that $\delta \in \mathrm{Aut}(X)$.

By Theorem 1.7.2,

$$Y \cong \mathop{\mathbf{P}^*}_{m=1}^{\infty} \mathbb{Z}(p^{k_m}),$$

where $k_m \leq k_{m+1}$, $m = 1, 2, \dots$. We will denote elements of the group Y by $l = (l_m)_{m=1}^{\infty}$, where $l_m \in \mathbb{Z}(p^{k_m})$ and $l_m = 0$ for all but a finite set of indices. Let $\tilde{y} = (y_m)_{m=1}^{\infty} \in Y$ be an element such that $y_1 \neq 0$ and $y_m = 0$ for $m > 1$. We will prove that the automorphism δ and the element $\tilde{y}$ satisfy the conditions of Lemma 13.20.

Let $j \geq i$. It is easy to see that the homomorphism $\tilde{\pi}_{ij} : \mathbb{Z}(p^{k_i}) \mapsto \mathbb{Z}(p^{k_j})$ is defined by

$$\tilde{\pi}_{ij} t_i = p^{k_j - k_i} t_i, \quad t_i \in \mathbb{Z}(p^{k_i}).$$

A direct verification shows that the homomorphism $\tilde{\delta}$ is of the form $\tilde{\delta}(l_m)_{m=1}^{\infty} = (h_m)_{m=1}^{\infty}$, where

$$h_m = \begin{cases} l_1 & \text{if } m = 1, \\ \tilde{\pi}_{m-1,m} l_{m-1} + l_m & \text{if } m \text{ is even}, \\ \tilde{\pi}_{m-2,m} l_{m-2} + \tilde{\pi}_{m-1,m} l_{m-1} + l_m & \text{if } m \text{ is odd}, m \neq 1. \end{cases} \tag{13.38}$$

We conclude from (13.31) that $I - \delta$ is an epimorphism. Then by Theorem 1.13 (b), $I - \tilde{\delta}$ is a monomorphism, i.e., 13.20 (i) is fulfilled. It follows from (13.38) that 13.20 (ii) is true. Obviously, 13.20 (iii) is also fulfilled. Since the group X is totally disconnected, by Proposition 3.6, $\Gamma(X) = D(X)$. Hence $\Gamma(X) * I(X) = I(X)$, and the statement of the lemma follows from Lemma 13.20. □

Remark 13.23. Let G be a closed subgroup of a group X. Let $\alpha_j, \beta_j \in \mathrm{Aut}(G)$ and assume that the automorphisms α_j, β_j can be extended to some topological automorphisms $\bar{\alpha}_j, \bar{\beta}_j$ of the group X. Assume that there exist independent random variables ξ_j with values in G and distributions $\mu_j \notin \Gamma(G) * I(G)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent. Obviously, we can consider ξ_j as independent random variables taking values in the group X. Furthermore, the linear forms $L_1 = \bar{\alpha}_1\xi_1 + \cdots + \bar{\alpha}_n\xi_n$ and $L_2 = \bar{\beta}_1\xi_1 + \cdots + \bar{\beta}_n\xi_n$ will be independent and $\mu_j \notin \Gamma(X) * I(X)$.

We also note that if G is a topological direct factor of X, then any topological automorphism $\delta \in \mathrm{Aut}(G)$ can be extended to a topological automorphism of the group X.

Lemma 13.24. *Let X be a compact totally disconnected group. Then either the group X is topologically isomorphic to a group of the form*

$$\text{(i)} \quad \mathop{\mathbf{P}}_{p \in \mathcal{P}} (\Delta_p^{n_p} \times G_p),$$

where n_p is a nonnegative integer and G_p is a finite p-primary group, or for some prime number p there exists a topological direct factor K of the group X such that K is topologically isomorphic to either the group $\Delta_p^{\aleph_0}$ or the group

$$\text{(ii)} \quad \mathop{\mathbf{P}}_{n=1}^{\infty} \mathbb{Z}(p^{m_n}), \quad m_n \leq m_{n+1}, \quad n = 1, 2, \ldots.$$

Proof. By Theorems 1.6.1 and 1.6.4, Y is a discrete torsion group. We deduce from Theorem 1.19.1 that Y is a weak direct product of its p-components Y_p,

$$Y = \mathop{\mathbf{P}^*}_{p \in \mathcal{P}} Y_p. \tag{13.39}$$

By Theorem 1.19.2 each p-primary subgroup Y_p can be represented as a direct product $Y_p = D_p \times N_p$, where D_p is the maximal divisible subgroup of Y_p and N_p is a countable reduced p-primary group. By Theorem 1.19.3 the group D_p can be represented in its turn as a weak direct product of groups each of which is isomorphic to the group $Z(p^\infty)$. Taking into account that $(Z(p^\infty))^* \cong \Delta_p$, we conclude from Theorem 1.7.2 that

$$X = \mathop{\mathbf{P}}_{p \in \mathcal{P}} X_p,$$

where

$$X_p \cong \Delta_p^{\mathfrak{n}} \times G_p, \quad \mathfrak{n} \le \aleph_0,\ G_p \cong (N_p)^*. \tag{13.40}$$

Assume that the group X is not topologically isomorphic to a group of the form (i). This means that in (13.40) for some p either $\mathfrak{n} = \aleph_0$ or G_p is an infinite group. If $\mathfrak{n} = \aleph_0$, then the group X has a topological direct factor G topologically isomorphic to the group $\Delta_p^{\aleph_0}$. If G_p is an infinite group, then the group N_p is also infinite. As is well known every countable infinite p-primary group has a direct factor isomorphic to the group

$$\mathop{\mathbf{P}^*}_{n=1}^{\infty} \mathbb{Z}(p^{m_n}), \quad m_n \le m_{n+1},\ n = 1, 2, \dots$$

([51], Proposition 77.5). It follows from this that the group N_p also has such a direct factor. But then by Theorem 1.7.2 the group G_p has a topological direct factor G, topologically isomorphic to a group of the form (ii). □

Theorem 13.25. *Let X be a compact totally disconnected group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of X. The independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$ if and only if X is topologically isomorphic to a group of the form* 13.24 (i).

Proof. We first prove the necessity. Assume that a group X is not topologically isomorphic to a group of the form 13.24 (i). By Lemma 13.24 for some prime number p the group X has a topological direct factor G such that either G is topologically isomorphic to the group $\Delta_p^{\aleph_0}$ or to a group of the form 13.24 (ii). If G is topologically isomorphic to the group $\Delta_p^{\aleph_0}$, then by Lemma 13.21 there exist an automorphism $\delta \in \mathrm{Aut}(G)$ and independent random variables ξ_1 and ξ_2 with values in G and distributions $\mu_1, \mu_2 \notin I(G)$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. If G is topologically isomorphic to a group of the form 13.24 (ii), then by Lemma 13.22 the analogous statement is also true. Since the subgroup G is a topological direct factor of X, the necessity results from Remark 13.23.

We prove the sufficiency. Assume that a group X is topologically isomorphic to a group of the form 13.24 (i). By Lemma 10.1 if the linear forms L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i). Set $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \ge 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation 10.1 (i). If we prove that $\nu_j \in I(X)$, then 2.7 (b) and 2.7 (e) imply that $\mu_j \in I(X)$. Thus we can solve equation 10.1 (i) assuming that $\hat\mu_j(y) \geq 0$. The sufficiency will be proved if we show that the functions $\hat\mu_j(y)$ take only the values 0 and 1. Indeed, in this case $\hat\mu_j^2(y) = \hat\mu_j(y)$, $y \in Y$, $j = 1, 2$. Then it follows from 2.7 (b) and 2.7 (c) that $\mu_j^{*2} = \mu_j$, i.e., $\mu_j \in I(X)$, $j = 1, 2$.

Let $y_0 \in Y$. Taking into account (13.39) we have

$$y_0 = \sum_{j=1}^{n} y_j,$$

where $y_j \in Y_{p_j}$. It is obvious that $p_j^{k_j} y_j = 0$, $j = 1, 2, \dots, n$ for some natural k_j. Consider the subgroups $B_j = \{y \in Y_{p_j} : {p_j}^{k_j} y = 0\}$. Then $y_0 \in B = B_1 \times \cdots \times B_n$. Since the group X is topologically isomorphic to a group of the form 13.24 (i), every subgroup B_j is finite. Hence the subgroup B is also finite. Obviously, B is a characteristic subgroup of the group Y. It is clear that for any $u, v \in Y$ there exists a subgroup B of the form above such that $u, v \in B$. Consider the restriction of equation 10.1 (i) to the subgroup B. By Corollary 2.11 the restriction of the characteristic functions $\hat\mu_j(y)$ to B are the characteristic functions of some distributions on the factor group $X/A(X, B)$. Note that by Theorems 1.9.1 and 1.9.2, $(X/A(X, B))^* \cong B$. Taking into account that the group B is finite, this implies that the factor group $X/A(X, B)$ is also finite. By Theorem 13.1 the characteristic functions $\hat\mu_j(y)$ take only values 0 and 1. The sufficiency is also proved. □

Consider now the case when X is a compact connected group. The following theorem is valid.

Theorem 13.26. *Let X be a compact connected group. Then there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ and independent random variables ξ_1 and ξ_2 with values in the group X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent.*

Proof. By Theorems 1.6.1 and 1.6.2, Y is a discrete torsion-free group. Two cases are possible: $f_p \notin \mathrm{Aut}(X)$ for some prime number p and $f_p \in \mathrm{Aut}(X)$ for all prime numbers p.

1. Suppose that

$$f_p \notin \mathrm{Aut}(X) \tag{13.41}$$

for some prime number p. We will assume that p in (13.41) is the smallest one. Since X is a connected group, by Theorem 1.9.6, $X^{(n)} = X$ for all $n \in \mathbb{N}$. Therefore, if $f_p \notin \mathrm{Aut}(X)$, then $\mathrm{Ker}\, f_p \neq \{0\}$.

Let $p = 2$. Then $\mathrm{Ker}\, f_2 = \{x \in X : 2x = 0\} \neq \{0\}$, i.e., the group X contains elements of order 2. Since $c_X = X$, by Theorem 7.10 there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent.

Assume that $p \geq 3$. Set $a = 1 - p$. Since p is the smallest natural number with property (13.41), we have $f_{-a} \in \mathrm{Aut}(X)$. Hence $f_a \in \mathrm{Aut}(X)$. By Theorem 1.9.5, $\mathrm{Ker}\, f_p = A(X, Y^{(p)})$. This implies that $Y^{(p)} \neq Y$. Take $\tilde{y} \notin Y^{(p)}$ and verify that the automorphism $\delta = f_a$ and the element $\tilde{y}$ satisfy the conditions of Lemma 13.20. We conclude from 1.13 (d) that $\tilde{f_a} = f_a$. We have $I - \tilde{f_a} = \tilde{f_p}$. Since Y is a torsion-free group, $\mathrm{Ker}(I - \tilde{f_a}) = \{0\}$, i.e., 13.20 (i) is fulfilled. It follows from $I - \tilde{f_a} = \tilde{f_p}$ that $(I - \tilde{f_a})Y = Y^{(p)}$. Since $p \geq 3$, the numbers 2 and p are relatively prime. Therefore there exist integers m and n such that $2m + pn = 1$. It follows from this that $y = 2my + pny$. Hence if $\tilde{y} \notin Y^{(p)}$, then $2\tilde{y} \notin Y^{(p)}$. This implies that 13.20 (ii) is true. Since Y is a torsion-free group, it is obvious that 13.20 (iii) is fulfilled. In this case the theorem follows from Lemma 13.20.

2. $f_p \in \mathrm{Aut}(X)$ for all prime numbers p. This means that X is a torsion-free group. Since X is a connected group, by Theorem 1.11.4 the group X is topologically isomorphic to a group of the form

$$(\Sigma_{\boldsymbol{a}})^{\mathfrak{n}}, \quad \boldsymbol{a} = (2, 3, 4, \dots),\ \mathfrak{n} \leq \aleph_0.$$

It is clear that it suffices to prove the theorem for the group $X = \Sigma_{\boldsymbol{a}}$. Then $Y \cong \mathbb{Q}$. We will denote elements of the group Y by r, $r \in \mathbb{Q}$.

Let H be a subgroup of Y of the form $H = \{\frac{m}{5^n}\}_{m,n \in \mathbb{Z}}$. Set $G = H^*$, $K = A(G, H^{(2)})$. It follows from Theorems 1.9.1 and 1.9.2 that $K \cong \mathbb{Z}(2)$. Let λ be an arbitrary non-idempotent distribution on the group G supported in K. It is easy to see that the characteristic function $\hat{\lambda}(h)$ is of the form

$$\hat{\lambda}(h) = \begin{cases} 1 & \text{if } h \in H^{(2)}, \\ c & \text{if } h \notin H^{(2)}, \end{cases} \tag{13.42}$$

where $-1 < c < 1$. Consider on the group Y the function

$$g(y) = \begin{cases} \hat{\lambda}(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases} \tag{13.43}$$

By Proposition 2.12, $g(y)$ is a positive definite function. By the Bochner theorem there exists a distribution $\mu \in \mathrm{M}^1(X)$ such that $\hat{\mu}(y) = g(y)$. It is obvious that $\mu \notin \Gamma(X) * I(X)$.

Let ξ_1 and ξ_2 be independent identically distributed random variables with values in X and distribution μ. We will verify that the linear forms $L_1 = \xi_1 + 2\xi_2$ and $L_2 = 2\xi_1 - \xi_2$ are independent. By Lemma 10.1 it suffices to show that the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$\hat{\mu}(u + 2v)\hat{\mu}(2u - v) = \hat{\mu}(u)\hat{\mu}(2u)\hat{\mu}(2v)\hat{\mu}(-v), \quad u, v \in Y. \tag{13.44}$$

It follows from (13.42) that if $u, v \in H$, then equation (13.44) is satisfied. It is also obvious that equation (13.44) holds if either $u \in H$, $v \notin H$ or $v \in H$, $u \notin H$.

Assume that $u, v \notin H$. Then the right-hand side of equation (13.44) is equal to zero. If the left-hand side of equation (13.44) is not equal to zero, then $u + 2v, 2u - v \in H$. This implies that $5u \in H$, and hence $u \in H$. The contradiction obtained shows that equation (13.44) is true for all $u, v \in Y$. □

Remark 13.27. Let X be a compact connected group. Assume that automorphisms $\alpha_j = f_{m_j}, \beta_j = f_{n_j} \in \mathrm{Aut}(X)$ satisfy the condition

$$\alpha_1\beta_1 + \alpha_2\beta_2 = 0. \tag{13.45}$$

Then there exist independent identically distributed random variables ξ_1 and ξ_2 with values in X and nondegenerate Gaussian distribution γ such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent. A natural problem arises: is it possible to construct automorphisms α_j, β_j in Theorem 13.26 satisfying condition (13.45) and independent identically distributed random variables ξ_1 and ξ_2 with values in X and distribution $\mu \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent?

In case 2 this has already been done in the proof of Theorem 13.26.

Consider case 1. Let $f_p \notin \mathrm{Aut}(X)$ for some prime number p, where $p \geq 3$. Assume that ξ_1 and ξ_2 are independent identically distributed random variables with values in X and distribution μ. Consider the linear forms $L_1 = \xi_1 + a\xi_2$ and $L_2 = a\xi_1 - \xi_2$, where $a = p - 1$. We will show that there exists a distribution $\mu \notin \Gamma(X) * I(X)$ such that the linear forms L_1 and L_2 are independent. Set $q = a^2 + 1$. By Lemma 10.1 the independence of L_1 and L_2 is equivalent to the fact that the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$\hat{\mu}(u + av)\hat{\mu}(au - v) = \hat{\mu}(u)\hat{\mu}(au)\hat{\mu}(av)\hat{\mu}(-v), \quad u, v \in Y. \tag{13.46}$$

If $f_q \notin \mathrm{Aut}(X)$, we reason as in case 1 of the proof of Theorem 13.26. Take $\tilde{y} \notin Y^{(q)}$. Let μ be the distribution on the group X with density $\rho(x) = 1 + \mathrm{Re}(x, \tilde{y})$ with respect to the Haar distribution m_X. It is clear that $\mu \notin \Gamma(X) * I(X)$. We verify that the characteristic function $\hat{\mu}(y)$ satisfies equation (13.46). It is obvious that equation (13.46) holds if either $u = 0$ or $v = 0$. Let $u \neq 0$ and $v \neq 0$. Since $a\tilde{y} \neq -\tilde{y}$, the right-hand side of equation (13.46) is equal to zero for all $v \neq 0$. If the left-hand side of equation (13.46) is not equal to zero, this implies that $u + av, au - v \in \{0, \pm\tilde{y}\}$. It follows from this that

$$qv \in \{0, \pm\tilde{y}, \pm(a \pm 1)\tilde{y}\}. \tag{13.47}$$

By the condition $\pm\tilde{y} \notin Y^{(q)}$. Taking into account that the numbers $a + 1$ and q are relatively prime and the numbers $a - 1$ and q are also relatively prime, we have $(a \pm 1)\tilde{y} \notin Y^{(q)}$. Therefore it follows from (13.47) that $qv = 0$. Since Y is a torsion-free group, $v = 0$. The contradiction obtained proves that the left-hand side of equation (13.46) is also equal to zero.

If $f_q \in \mathrm{Aut}(X)$, we follow the scheme of the proof of Theorem 13.26 in case 2. Namely, let y_0 be an arbitrary element of the group Y. Since $f_q \in \mathrm{Aut}(X)$, we have $f_q \in \mathrm{Aut}(Y)$. Therefore we can consider a subgroup H of the group Y of the form

$H = \{\frac{m}{q^n} y_0\}_{m,n \in \mathbb{Z}}$. Let $G = H^*$. Since the numbers q and a are relatively prime, $H \neq H^{(a)}$. Consider the distribution $\pi = \gamma * \lambda \in \mathrm{M}^1(G)$, where $\gamma \in \Gamma(G)$, $\lambda \in \mathrm{M}^1(K)$, $K = A(G, H^{(a)})$ and construct the distribution μ as has already been done in Theorem 13.26 in case 2.

If we assume that $f_2 \notin \mathrm{Aut}(X)$ for a group X, then the answer to the above question, generally, is negative. Indeed, let $X = \mathbb{T}$. Then $\mathrm{Aut}(\mathbb{T}) = \{\pm I\}$, and for this reason the study of arbitrary linear forms L_1 and L_2 is reduced to the forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$. Since the group X contains only one element of order 2, by Theorem 9.9 if ξ_1 and ξ_2 are independent identically distributed random variables with values in X and distribution μ such that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent, then $\mu \in \Gamma(X) * I(X)$.

On the other hand, let $X = \mathbb{T}^2$. As follows from Lemma 9.6, there exist independent identically distributed random variables ξ_1 and ξ_2 with values in X and distribution $\mu \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent.

Remark 13.28. Let X be a compact connected group. Following the scheme of the proof of Theorem 13.26 we will verify that the statement of Theorem 13.26 is true for an arbitrary number $n \geq 2$ of independent random variables.

Let $f_2 \notin \mathrm{Aut}(X)$. Consider the distributions μ_1 and μ_2 on the circle group $\mathbb{T}$ constructed in Lemma 7.8. We conclude from 2.7 (c) that the parameters a and b in (7.11) can be chosen in such a way that $\mu_1 = \pi^{*(n-1)}$, where $\pi \in \mathrm{M}^1(\mathbb{T})$. Let ξ_j be independent random variables with values in the circle group $\mathbb{T}$ and distributions ν_j, where $\nu_1 = \cdots = \nu_{n-1} = \pi$, $\nu_n = \mu_2$. It follows from Lemmas 7.8 and 10.1 that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-1} - \xi_n$ are independent. Next we argue as in the proof of statement (II) of Theorem 7.10 and get the desired statement.

If $f_p \notin \mathrm{Aut}(X)$ for some prime number $p \geq 3$, then the required assertion follows directly from Lemma 13.20.

Let $f_p \in \mathrm{Aut}(X)$ for all prime numbers p. Arguing as in Theorem 13.26, we assume that $X = \Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (2, 3, 4, \ldots)$. Let μ be a distribution on the group X with the characteristic function defined by (13.43). Let ξ_j be independent identically distributed random variables with values in X and distribution μ. Arguing as in the proof of Theorem 13.26 and using Lemma 10.1, we show that the linear forms $L_1 = \xi_1 + \cdots + \xi_{n-1} + \xi_n$ and $L_2 = 2\xi_1 + \cdots + 2\xi_{n-1} - \xi_n$ are independent.

14 The number of random variables $n \geq 3$

Let X be a second countable locally compact Abelian group, Y be its character group, ξ_j, $j = 1, 2, \ldots, n$, $n \geq 3$, be independent random variables with values in X and distributions μ_j. Let $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$, where the coefficients α_j, $\beta_j \in \mathrm{Aut}(X)$. It turns out that in contrast to the case $n = 2$ the classes of groups for which the Skitovich–Darmois theorem is true are very poor. We study group analogues of the Skitovich–Darmois theorem for finite, compact totally disconnected, discrete torsion and compact groups. First we study the case when X is a finite group. We need some lemmas.

Lemma 14.1. *Let X be a group of the form*

$$\text{(i)}\quad \mathbb{Z}(2^{m_1})\times\cdots\times\mathbb{Z}(2^{m_l}),\quad \textit{where } 0\le m_1<\cdots<m_l,$$

let $\alpha_j,\beta_j\in\mathrm{Aut}(X)$, $j=1,2,\dots,n$, $n\ge 2$, and let ξ_j be independent random variables with values in X and distributions μ_j. If the linear forms $L_1=\alpha_1\xi_1+\cdots+\alpha_n\xi_n$ and $L_2=\beta_1\xi_1+\cdots+\beta_n\xi_n$ are independent, then all μ_j are degenerate distributions.

Proof. Arguing as in the proof of Theorem 10.3, we reduce the proof of the lemma to the case when $L_1=\xi_1+\cdots+\xi_n$ and $L_2=\delta_1\xi_1+\cdots+\delta_n\xi_n$, where $\delta_j\in\mathrm{Aut}(X)$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat\mu_j(y)$ satisfy equation 10.1 (i) which takes the form (10.2). Put $\nu_j=\mu_j*\bar\mu_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat\nu_j(y)=|\hat\mu_j(y)|^2\ge 0$, $y\in Y$. Obviously, the characteristic functions $\hat\nu_j(y)$ also satisfy equation (10.2). In view of 2.7 (b) and 2.7 (e) the lemma will be proved if we show that all $\nu_j\in D(X)$. Therefore, we can assume from the beginning that all $\hat\mu_j(y)\ge 0$, $y\in Y$. We will prove that in this case $\hat\mu_j(y)\equiv 1$, $j=1,2,\dots,n$.

By Theorem 1.7.1, $Y\cong X$. Let ε be an arbitrary automorphism of the group Y. Put $H_i=Y^{(2^{m_i-1})}\cap Y_{(2)}$, $i=1,2,\dots,l$. Since for any natural k the subgroups $Y^{(k)}$ and $Y_{(k)}$ are characteristic, the subgroups H_i are also characteristic. This implies that

$$\varepsilon(H_{i-1}\backslash H_i)=H_{i-1}\backslash H_i. \tag{14.1}$$

It follows from (14.1) that if $u,v\in H_{i-1}\backslash H_i$, then $u+\varepsilon v\in H_i$, $i=1,2,\dots,l$. It is easy to see that $H_l\cong\mathbb{Z}(2)$. Hence $\varepsilon y=y$, $y\in H_l$. The restriction of equation (10.2) to the subgroup H_l takes the form

$$\prod_{j=1}^n\hat\mu_j(u+v)=\prod_{j=1}^n\hat\mu_j(u)\prod_{j=1}^n\hat\mu_j(v),\quad u,v\in H_l. \tag{14.2}$$

Put

$$f(y)=\prod_{j=1}^n\hat\mu_j(y),\quad y\in Y.$$

We deduce from (14.2) that the restriction of the function $f(y)$ to the subgroup H_l is a character of the group H_l. Taking into account that $f(y)\ge 0$, we get that $f(y)=1$, $y\in H_l$. Hence $\hat\mu_j(y)=1$, $y\in H_l$, $j=1,2,\dots,n$. Substituting $u,v\in H_{l-1}\backslash H_l$ into equation (10.2) and taking into account that $u+\tilde\delta_j v\in H_l$ for all $\tilde\delta_j$, we obtain that $\hat\mu_j(y)=1$, $y\in H_{l-1}$, $j=1,2,\dots,n$. Repeating this procedure, we prove that $\hat\mu_j(y)=1$, $y\in Y_{(2)}$, $j=1,2,\dots,n$.

Let us prove now that $\hat{\mu}_j(y) = 1$, $y \in Y$, $j = 1, 2, \ldots, n$. We will prove this by induction on m_l. For $m_l = 1$ we have $X = \mathbb{Z}(2)$. Then $Y \cong \mathbb{Z}(2)$ and hence $Y = Y_{(2)}$. As has been noted above, if $y \in Y_{(2)}$, then $\hat{\mu}_j(y) = 1$, $j = 1, 2, \ldots, n$. Thus for $m_l = 1$ the statement is true.

Since $\hat{\mu}_j(y) = 1$, $y \in Y_{(2)}$, $j = 1, 2, \ldots, n$, by Proposition 2.13 the functions $\hat{\mu}_j(y)$ take constant values on each coset $y + Y_{(2)}$. Hence they induce some characteristic functions $g_j([y])$ on the factor group $Y/Y_{(2)}$ by the formula $g_j([y]) = \hat{\mu}_j(y)$, $y \in [y]$. Since $Y_{(2)}$ is a characteristic subgroup of the group Y, the automorphisms $\tilde{\delta}_j$ also induce some automorphisms ε_j on the factor group $Y/Y_{(2)}$ by the formula $\varepsilon_j[y] = [\tilde{\delta}_j y]$, $y \in [y]$. It means that we can consider equation (10.2) on the factor group $Y/Y_{(2)}$. Taking into account that

$$Y/Y_{(2)} \cong \mathbb{Z}(2^{m_1-1}) \times \cdots \times \mathbb{Z}(2^{m_l-1}),$$

by the induction hypothesis $g_j([y]) = 1$, $j = 1, 2, \ldots, n$, for all $[y] \in Y/Y_{(2)}$. Hence $\hat{\mu}_j(y) = 1$, $y \in Y$, $j = 1, 2, \ldots, n$. □

Corollary 14.2. *Let Y be a group of the form* 14.1 (i) *and $\tilde{\alpha}_j, \tilde{\beta}_j \in \mathrm{Aut}(Y)$, $j = 1, 2, \ldots, n$, $n \geq 2$. Let $\hat{\mu}_j(y)$ be characteristic functions on the group Y satisfying equation* 10.1 (i). *Then $\hat{\mu}_j(y)$ are of form*

$$\hat{\mu}_j(y) = (x_j, y), \quad x_j \in X, \ j = 1, 2, \ldots, n.$$

Lemma 14.3. *Let $X = \mathbb{Z}(3)$, $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \ldots, n$, $n \geq 2$. Let ξ_j be independent random variables with values in X and distributions μ_j. If the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, then either all μ_j are degenerate distributions or $\mu_{j_1} = \mu_{j_2} = m_X$ for at least two distributions μ_{j_1} and μ_{j_2}, and the remaining μ_j are arbitrary distributions.*

Proof. Note that $\mathrm{Aut}(X) = \{\pm I\}$. Obviously, we can assume without loss of generality that $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_m - \cdots - \xi_n$. Consider new independent random variables $\eta_1 = \xi_1 + \cdots + \xi_m$ and $\eta_2 = \xi_{m+1} + \cdots + \xi_n$. By the condition of the lemma the linear forms $L_1 = \eta_1 + \eta_2$ and $L_2 = \eta_1 - \eta_2$ are independent. Denote by λ_j the distribution of the random variables η_j. We conclude from (2.1) that $\lambda_1 = \mu_1 * \cdots * \mu_m$, $\lambda_2 = \mu_{m+1} * \cdots * \mu_n$. Since $c_X = \{0\}$, by Theorem 7.10 either $\lambda_1, \lambda_2 \in D(X)$ or $\lambda_1 = \lambda_2 = m_X$. If $\lambda_1, \lambda_2 \in D(X)$, then all $\mu_j \in D(X)$. It follows directly from 2.7 (c) and 2.14 (i) that if $X = \mathbb{Z}(3)$ and $\pi_1 * \pi_2 = m_X$, then $\pi_j = m_X$ at least for one distribution π_j. Hence if $\lambda_1 = \lambda_2 = m_X$, then $\mu_{j_1} = \mu_{j_2} = m_X$ at least for two distributions μ_{j_1} and μ_{j_2}, and the remaining μ_j are arbitrary distributions. □

Lemma 14.4. *Let $X = \mathbb{Z}(5)$, α_j, $\beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, 3$. Let ξ_j be independent random variables with values in X and distributions μ_j. If the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3$ are independent, then either all μ_j are degenerate distributions or $\mu_{j_1} = m_X$ for at least one distribution μ_{j_1}.*

Proof. Note that $Y \cong \mathbb{Z}(5)$. Arguing as in the proof of Theorem 10.3, we reduce the proof of the lemma to the case when $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2 + \delta_3\xi_3$, where $\delta_j \in \mathrm{Aut}(X)$.

First assume that not all automorphisms δ_j are different. Let for definiteness $\delta_2 = \delta_3 = \delta$. Consider the random variables ξ_1 and $\xi = \xi_2 + \xi_3$. Then the linear forms $L_1 = \xi_1 + \xi$ and $L_2 = \delta_1\xi_1 + \delta\xi$ are independent. By (2.1) the random variable ξ has the distribution $\mu = \mu_2 * \mu_3$. It follows from Theorem 13.1 that either μ_1 and μ are degenerate distributions and then all μ_j are degenerate distributions, or $\mu_1 = \mu = m_X$.

Assume now that all automorphisms δ_j are different. Obviously, every automorphism $\delta \in \mathrm{Aut}(X)$ is of the form $\delta x = kx$, $k = 1, 2, 3, 4$, $x \in X$. Since the linear forms L_1 and L_2 are independent if and only if the linear forms L_1 and δL_2 are independent, it is easy to see that we can assume without loss of generality that $L_2 = \xi_1 + 2\xi_2 + 3\xi_3$. Set $f(y) = \hat\mu_1(y)$, $g(y) = \hat\mu_2(y)$, $h(y) = \hat\mu_3(y)$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat\mu_j(y)$ satisfy equation 10.1 (i) which takes the form

$$f(u+v)g(u+2v)h(u+3v) = f(u)g(u)h(u)f(v)g(2v)h(3v), \quad u, v, \in Y. \tag{14.3}$$

We note that the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Therefore, if the characteristic functions $\hat\mu_j(y)$ do not vanish, then by Theorem 10.3 all μ_j are degenerate distributions.

Suppose that only one characteristic function $\hat\mu_j(y)$ vanishes. We note that for any $u \in Y$, $u \neq 0$, and any automorphism $\varepsilon \in \mathrm{Aut}(Y)$ there exists an element $v \in Y$ such that $u + \varepsilon v = 0$. Assume for definiteness that $f(y)$ vanishes and $g(y)$ and $h(y)$ do not vanish. Let $f(u_0) = 0$. Substitute $u = u_0$, $v = -u_0$ into equation (14.3). Then the right-hand side of equation (14.3) is equal to zero, whereas the left-hand side is not. The contradiction obtained shows that at least two characteristic functions vanish.

Assume that only two characteristic functions, say for definiteness $f(y)$ and $g(y)$ vanish. Then they have common zeros. Indeed, in the opposite case we have $f(u_0) = g(2u_0) = 0$, $f(2u_0) \neq 0$, $g(u_0) \neq 0$ for some $u_0 \in Y$. Hence for $u = 3u_0$, $v = 4u_0$ the right-hand side of equation (14.3) is equal to zero, whereas the left-hand side is not. Thus $f(y)$ and $g(y)$ have common zeros. Let $f(u_0) = g(u_0) = 0$. Substituting $u = u_0$, $v = 2u_0$ into (14.3) we get $f(3u_0) = 0$, i.e., $\mu_1 = m_X$. Substituting $u = 3u_0$, $v = 2u_0$ into (14.3) we get $g(2u_0) = 0$, i.e., $\mu_2 = m_X$. The other cases, where either $f(y)$ and $h(y)$ vanish or $g(y)$ and $h(y)$ vanish can be considered similarly.

Assume now that all three characteristic functions $f(y)$, $g(y)$, and $h(y)$ vanish. We first consider the case when, at a certain point $u_0 \in Y$,

$$f(u_0) = g(u_0) = h(u_0) = 0. \tag{14.4}$$

Putting $u = u_0$, $v = 2u_0$ in (14.3) we obtain $f(3u_0)h(2u_0) = 0$. This implies that either $\mu_1 = m_X$ or $\mu_3 = m_X$.

If (14.4) is not true, then the following three cases are possible:

1. $f(u_0) = g(u_0) = h(2u_0) = 0$, $h(u_0) \neq 0$ at a certain point $u_0 \in Y$. Putting $u = 2u_0$, $v = 3u_0$ in (14.3) we get $g(3u_0) = 0$, i.e., $\mu_2 = m_X$.

2. $f(u_0) = g(2u_0) = h(u_0) = 0$, $g(u_0) \neq 0$ at a certain point $u_0 \in Y$. Substituting $u = 2u_0$, $v = u_0$ into (14.3) we obtain $f(3u_0) = 0$, i.e., $\mu_1 = m_X$. Putting $u = u_0$, $v = 4u_0$ in (14.3) we get $h(3u_0) = 0$, i.e., $\mu_3 = m_X$.

3. $f(2u_0) = g(u_0) = h(u_0) = 0$, $f(u_0) \neq 0$ at a certain point $u_0 \in Y$. Substituting $u = 4u_0$, $v = 2u_0$ into (14.3) we obtain $g(3u_0) = 0$, i.e., $\mu_2 = m_X$. □

Remark 14.5. It should be noted that Lemma 14.4 may not be strengthened. To put it in another way: it is not true that $\mu_{j_1} = \mu_{j_2} = m_X$ for at least two distributions μ_{j_1} and μ_{j_2}. To prove this take $u_0 \in Y$, $u_0 \neq 0$, and consider the distributions $\mu_1 = m_X$, $\mu_2(\{x\}) = (1/5)(1 + \mathrm{Re}(x, u_0))$, $x \in X$, and $\mu_3 = \mu_2$. Then $\hat{\mu}_j(y) = 0$ for $y \in \{2u_0, 3u_0\}$, $j = 2, 3$. It is easy to check that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (14.3). Hence by Lemma 10.1 if ξ_j are independent random variables with values in X and distributions μ_j, then the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_1 = \xi_1 + 2\xi_2 + 3\xi_3$ are independent.

Lemma 14.6. *For each of the groups $X = (\mathbb{Z}(k))^2$, where $k \geq 2$, $X = \mathbb{Z}(2k-1)$, where $k \geq 4$, and $X = \mathbb{Z}(3) \times \mathbb{Z}(5)$, and for every $n \geq 3$ there exist automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ and independent identically distributed random variables ξ_j, $j = 1, 2, \dots, n$, with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \dots + \xi_n$ and $L_2 = \xi_1 + \dots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ are independent.*

Proof. We note that $Y \cong X$. Let $X = (\mathbb{Z}(k))^2$, where $k \geq 2$. Denote by (p, q), $p, q \in \mathbb{Z}(k)$, elements of the groups X and Y. Set $\tilde{y} = (0, 1) \in Y$. Let μ be the distribution on the group X such as in Lemma 13.20. Then the characteristic function $f(y) = \hat{\mu}(y)$ is defined by (13.30). It is obvious that $\mu \notin I(X)$. Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 3$, be independent identically distributed random variables with values in X and distribution μ. Let automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ be of the form

$$\alpha(p, q) = (p, p + q), \quad \beta(p, q) = (q, p), \quad p, q \in \mathbb{Z}(k). \tag{14.5}$$

Then

$$\tilde{\alpha}(p, q) = (p + q, q), \quad \tilde{\beta}(p, q) = (q, p), \quad p, q \in \mathbb{Z}(k). \tag{14.6}$$

We verify that the linear forms $L_1 = \xi_1 + \dots + \xi_n$ and $L_2 = \xi_1 + \dots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ are independent. By Lemma 10.1 it suffices to show that the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$\begin{aligned} f^{n-2}(u + v) f(u + \tilde{\alpha} v) f(u + \tilde{\beta} v) \\ = f^n(u) f^{n-2}(v) f(\tilde{\alpha} v) f(\tilde{\beta} v), \quad u, v \in Y. \end{aligned} \tag{14.7}$$

Obviously, equation (14.7) is satisfied if either $u = 0$ or $v = 0$. Let $u \neq 0$ and $v \neq 0$. Since $\tilde{\beta}\tilde{y} \neq \pm\tilde{y}$, the right-hand side of equation (14.7) is equal to zero. If the left-hand side of equation (14.7) is not equal to zero, then as follows from (13.30) we have $u + v,\ u + \tilde{\alpha} v,\ u + \tilde{\beta} v \in \{0, \pm\tilde{y}\}$. We conclude from this that

$$(\tilde{\alpha} - I)v \in \{0, \pm\tilde{y}, \pm 2\tilde{y}\} \tag{14.8}$$

and

$$(\tilde{\alpha} - \tilde{\beta})v \in \{0, \pm\tilde{y}, \pm 2\tilde{y}\}. \tag{14.9}$$

Let $v = (p, q)$. Then $(\tilde{\alpha} - I)v = (q, 0)$. Hence (14.8) implies that $q = 0$. On the other hand, $(\tilde{\alpha} - \tilde{\beta})v = (p, q - p)$. Therefore, (14.9) yields that $p = 0$, and hence $v = 0$ contrary to the assumption. Thus the left-hand side of equation (14.7) is also equal to zero. We get that the characteristic function $f(y)$ satisfies equation (14.7). So, for the group $X = (\mathbb{Z}(k))^2$, $k \geq 2$, the lemma is proved.

Let either $X = \mathbb{Z}(2k-1)$, where $k \geq 4$ or $X = \mathbb{Z}(3) \times \mathbb{Z}(5)$. If $X = \mathbb{Z}(2k-1)$, where $k \geq 4$, then denote by $\tilde{y}$ an element of order $2k-1$ in Y. If $X = \mathbb{Z}(3) \times \mathbb{Z}(5)$, then denote by $\tilde{y}$ an element of order 15 in Y. Let μ be the distribution on the group X such as in Lemma 13.20. Then the characteristic function $f(y) = \hat{\mu}(y)$ is defined by (13.30). Obviously, $\mu \notin I(X)$. Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 3$, be independent identically distributed random variables with values in X and distribution μ. Let automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ be of the form $\alpha = f_2$, $\beta = -I$.

We check that the linear forms $L_1 = \xi_1 + \dots + \xi_n$ and $L_2 = \xi_1 + \dots + \xi_{n-2} + 2\xi_{n-1} - \xi_n$ are independent. By Lemma 10.1 it suffices to show that the characteristic functions of the random variables ξ_j satisfy equation 10.1 (i) which takes the form

$$f^{n-2}(u+v)f(u+2v)f(u-v) = f^n(u)f^{n-2}(v)f(2v)f(-v), \quad u, v \in Y. \tag{14.10}$$

Obviously, equation (14.10) holds if either $u = 0$ or $v = 0$. Let $u \neq 0$ and $v \neq 0$. Since $2\tilde{y} \neq \pm\tilde{y}$, the right-hand side of equation (14.10) is equal to zero. If the left-hand side of equation (14.10) is not equal to zero, then as follows from (13.30), we have

$$u \pm v, \ u + 2v \in \{0, \pm\tilde{y}\}. \tag{14.11}$$

This implies that

$$v, 2v \in \{0, \pm\tilde{y}, \pm 2\tilde{y}\}. \tag{14.12}$$

We conclude from (14.12) that $v = \pm\tilde{y}$. Taking into account that $u + v \in \{0, \pm\tilde{y}\}$, we have the following possibilities: $u = \tilde{y}, v = -\tilde{y}$; $u = 2\tilde{y}, v = -\tilde{y}$; $u = -\tilde{y}$, $v = \tilde{y}$; $u = -2\tilde{y}, v = \tilde{y}$. In each of these cases we have $u - v \notin \{0, \pm\tilde{y}\}$, contrary to (14.11). Thus the left-hand side of equation (14.10) is also equal to zero. We get that the characteristic function $f(y)$ satisfies equation (14.10). For the groups $X = \mathbb{Z}(2k-1)$, where $k \geq 4$, and $X = \mathbb{Z}(3) \times \mathbb{Z}(5)$ the lemma is also proved. □

Theorem 14.7. *Let X be a finite group, G be a group of the form* 14.1 (i). *Then the following statements hold:*

(I) *Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, 3$, and let ξ_j be independent random variables with values in X and distributions μ_j. Assume that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3$ are independent. If $X = G$, then all μ_j are degenerate distributions. If $X = \mathbb{Z}(3) \times G$, then either all μ_j are degenerate distributions or $\mu_{j_1} * E_{x_1} = \mu_{j_2} * E_{x_2} = m_{\mathbb{Z}(3)}$, $x_j \in X$, for at least two distributions μ_{j_1} and μ_{j_2}. If $X = \mathbb{Z}(5) \times G$, then either all μ_j are degenerate distributions or $\mu_{j_1} * E_{x_1} = m_{\mathbb{Z}(5)}$, $x_1 \in X$, for at least one distribution μ_{j_1}.*

(II) *If a group X is not isomorphic to any of the groups mentioned in* (I)*, then for every $n \geq 3$ there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, and independent identically distributed random variables ξ_j with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent.*

Proof. (I). If $X = G$, then the statement of the theorem follows from Lemma 14.1.

Let $X = \mathbb{Z}(3) \times G$. By Theorem 1.7.1, $Y = L \times H$, where $L \cong \mathbb{Z}(3)$, $H \cong G$. By Lemma 10.1 if the linear forms L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i). Put $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation 10.1 (i). Since H is the 2-component of the group Y, the subgroup H is characteristic. Consider the restriction of equation 10.1 (i) to H. Applying Corollary 14.2 we get that $\hat{\nu}_j(y) = 1$ for all $y \in H$. By Proposition 2.13 the inclusions $\sigma(\nu_j) \subset A(X, H) = \mathbb{Z}(3)$, $j = 1, 2, 3$ hold. It follows from Proposition 2.2 that the distributions μ_j can be replaced by their shifts μ'_j in such a manner that $\sigma(\mu'_j) \subset \mathbb{Z}(3)$. Let ξ'_j be independent random variables with values in X and distributions μ'_j. By Corollary 10.2 the linear forms $L'_1 = \alpha_1\xi'_1 + \alpha_2\xi'_2 + \alpha_3\xi'_3$ and $L'_2 = \beta_1\xi'_1 + \beta_2\xi'_2 + \beta_3\xi'_3$ are independent. Since $\mathbb{Z}(3)$ is a characteristic subgroup of the group X, we can suppose that $\alpha_j, \beta_j \in \mathrm{Aut}(\mathbb{Z}(3))$ and ξ'_j take values in the group $\mathbb{Z}(3)$. Then the statement of the theorem follows from Lemma 14.3.

For the group $X = \mathbb{Z}(5) \times G$ we reason similarly using Lemma 14.4 instead of Lemma 14.3.

(II). Assume now that a group X is not isomorphic to any of the groups mentioned in (I). By Theorem 1.19.1 we can represent the group X as a direct product of its p-components X_p, i.e.,

$$X = \mathop{\mathbf{P}}_{j=1}^{m} X_{p_j}.$$

By Theorem 1.19.4 each of the groups X_{p_j} is isomorphic to a finite direct product of groups of the form $(\mathbb{Z}({p_j}^m))^{k_m}$ for different m. It follows from this that if a group X is not isomorphic to any of the groups mentioned in (I), then the group X has a direct factor isomorphic to one of the groups mentioned in Lemma 14.6. Then the statement of the theorem follows from Lemma 14.6 and Remark 13.23. □

Remark 14.8. The proof of Theorem 14.7 in the case when $X = \mathbb{Z}(3) \times G$ is based on Lemma 14.3. Lemma 14.3 is valid for an arbitrary number $n \geq 2$ of independent random variables. Therefore the statement of Theorem 14.7 in the case when $X = \mathbb{Z}(3) \times G$ is also valid for an arbitrary number $n \geq 2$ of independent random variables.

Let us study now the case when X is a compact totally disconnected group.

Lemma 14.9. *Let either $X = \Delta_2$ or $X = \Delta_2 \times G$, where G is a group of the form* 14.1 (i)*. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, $n \geq 2$, and let ξ_j be independent random variables with values in X and distributions μ_j. If the linear forms*

$L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, then all μ_j are degenerate distributions.

Proof. By Lemma 10.1 it follows from the independence of the linear forms L_1 and L_2 that the characteristic functions $\hat\mu_j(y)$ satisfy equation 10.1 (i). Put $\nu_j = \mu_j * \bar\mu_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat\nu_j(y) = |\hat\mu_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat\nu_j(y)$ also satisfy equation 10.1 (i). The lemma will be proved if we show that all $\nu_j \in D(X)$. Therefore we can assume from the beginning that $\hat\mu_j(y) \geq 0$, $y \in Y$.

1. Let $X = \Delta_2$. We have $Y \cong \mathbb{Z}(2^\infty)$ and $Y_{(2^m)} \cong \mathbb{Z}(2^m)$. Note that the natural embeddings hold

$$Y_{(2)} \subset \cdots \subset Y_{(2^m)} \subset \cdots \subset Y, \tag{14.13}$$

and

$$Y = \bigcup_{m=1}^{\infty} Y_{(2^m)}. \tag{14.14}$$

Each of the subgroups $Y_{(2^m)}$ is characteristic. Consider the restriction of equation 10.1(i) to the subgroup $Y_{(2^m)}$. By Lemma 14.1, $\hat\mu_j(y) = 1$, $y \in Y_{(2^m)}$, $j = 1, 2, \ldots, n$. We deduce from (14.13) and (14.14) that $\hat\mu_j(y) = 1$, $y \in Y$, $j = 1, 2, \ldots, n$. The statement of the lemma follows from 2.7 (b).

2. Let $X = \Delta_2 \times G$, where G is a group of the form 14.1 (i). By Theorem 1.7.1, $Y = H \times L$, where $H \cong \mathbb{Z}(2^\infty)$, $L \cong G$. It follows from $G^{(2^{m_l})} = \{0\}$ that $L^{(2^{m_l})} = \{0\}$ and hence $H = Y^{(2^{m_l})}$. This implies that H is a characteristic subgroup of the group Y. Consider the restriction of equation 10.1 (i) to the subgroup H. As has been proved in case 1, $\hat\mu_j(y) = 1$, $y \in H$, $j = 1, 2, \ldots, n$, By Proposition 2.13 we conclude from this that the inclusions $\sigma(\mu_j) \subset A(X, H) = G$, $j = 1, 2, \ldots, n$. hold. Since H is a characteristic subgroup of the group Y and $G = A(X, H)$, by Lemma 13.11, G is a characteristic subgroup of the group X. Therefore α_j, β_j can be considered as automorphisms of the group G. We have independent random variables ξ_j with values in the group G and distributions μ_j such that linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent. The required statement follows from Lemma 14.1. □

Corollary 14.10. *Let either $Y = \mathbb{Z}(p^\infty)$ or $Y = \mathbb{Z}(p^\infty) \times H$, where H is a group of the form 14.1 (i). Let $\tilde\alpha_j, \tilde\beta_j \in \mathrm{Aut}(Y)$, $j = 1, 2, \ldots, n$, $n \geq 2$, and let $\hat\mu_j(y)$ be characteristic functions on the group Y satisfying equation 10.1 (i). Then $\hat\mu_j(y)$ are of the form*

$$\hat\mu_j(y) = (x_j, y), \quad x_j \in X, \ j = 1, 2, \ldots, n.$$

Lemma 14.11. *Let G be a compact subgroup of a group X, $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \ldots, n$, $n \geq 2$, and assume that the restrictions of automorphisms α_j, β_j to G are topological automorphisms of the group G. Let $g_j(y)$ be continuous normalized positive definite functions on the annihilator $A(Y, G)$. Assume that the following conditions are satisfied:*

(i) *the functions* $g_j(y)$ *satisfy equation* 10.1 (i);

(ii) *the characteristic functions* $\hat{\pi}_j(y) = \widehat{m}_G(y)$, $j = 1, 2, \ldots, n$, *satisfy equation* 10.1 (i) *when* $y \in Y$.

Let $\mu_j \in \mathrm{M}^1(X)$, $j = 1, 2, \ldots, n$, *and the characteristic functions* $\hat{\mu}_j(y)$, $y \in Y$, *be of the form*

$$\hat{\mu}_j(y) = \begin{cases} g_j(y) & \text{if } y \in A(Y, G), \\ 0 & \text{if } y \notin A(Y, G). \end{cases}$$

Then $\hat{\mu}_j(y)$ *also satisfy equation* 10.1 (i).

Proof. Set $L = A(Y, G)$. By Lemma 13.11 the restrictions of automorphisms $\tilde{\alpha}_j$, $\tilde{\beta}_j$ to L are topological automorphisms of the group L because the restrictions of automorphisms α_j, β_j to G are topological automorphisms of the group G. If $u, v \in L$, then in view of (i), equation 10.1 (i) holds. We also note that the characteristic function $\widehat{m}_G(y)$ is of the form 2.14 (i). Therefore, if either $u \in L$, $v \notin L$ or $u \notin L$, $v \in L$, then both sides of equation 10.1 (i) are equal to zero. If $u, v \notin L$, then the right-hand side of equation 10.1 (i) is equal to zero. If the left-hand side of equation 10.1 (i) is not equal to zero, then $\tilde{\alpha}_j u + \tilde{\beta}_j v \in L$, $j = 1, 2, \ldots, n$, contrary to (ii). □

Lemma 14.12. *For each of the groups* $X = \Delta_2^2$ *and* $X = \Delta_p$, *where* p *is a prime number,* $p \geq 3$, *and for every* $n \geq 3$ *there exist automorphisms* $\alpha, \beta \in \mathrm{Aut}(X)$ *and independent identically distributed random variables* ξ_j, $j = 1, 2, \ldots, n$, *with values in* X *and distribution* $\mu \notin I(X)$ *such that the linear forms* $L_1 = \xi_1 + \cdots + \xi_n$ *and* $L_2 = \xi_1 + \cdots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ *are independent.*

Proof. Let $X = \Delta_2^2$. Set $G = X^{(2)}$. Then G is a characteristic subgroup of the group X. By Theorem 1.7.1, $Y \cong (\mathbb{Z}(2^\infty))^2$, and by Theorem 1.9.5, $L = A(Y, G) = Y_{(2)}$. Obviously, $Y_{(2)} \cong (\mathbb{Z}(2))^2$. The subgroup L can be considered as the character group of the group $K = (\mathbb{Z}(2))^2$. Denote by (a, b), $a, b \in \Delta_2$, elements of the group X, and by (p, q), $p, q \in \mathbb{Z}(2^\infty)$, elements of the group Y. We will also denote by (a, b), $a, b \in \mathbb{Z}(2)$, elements of the group K, and by (p, q), $p, q \in \mathbb{Z}(2)$, elements of the group L. Let automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ be of the form

$$\alpha(a, b) = (a, a + b), \quad \beta(a, b) = (b, a), \quad a, b \in \Delta_2.$$

Then

$$\tilde{\alpha}(p, q) = (p + q, q), \quad \tilde{\beta}(p, q) = (q, p), \quad p, q \in \mathbb{Z}(2^\infty). \tag{14.15}$$

Let automorphisms $\alpha_1, \beta_1 \in \mathrm{Aut}(K)$ be defined by (14.5). We note that (14.6) and (14.15) imply that the restrictions of automorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ to the subgroup L coincide with the automorphisms $\tilde{\alpha}_1$ and $\tilde{\beta}_1$. By Lemma 14.6 there exist independent identically distributed random variables ζ_j, $j = 1, 2, \ldots, n$, $n \geq 3$, with values in the group K and distribution $\nu \notin I(K)$ such that the linear forms $L_1 = \zeta_1 + \cdots + \zeta_n$ and $L_2 = \zeta_1 + \cdots + \zeta_{n-2} + \alpha_1\zeta_{n-1} + \beta_1\zeta_n$ are independent. Then by Lemma 10.1 the characteristic functions of the random variables ζ_j satisfy equation 10.1 (i) which takes the form (14.7).

The characteristic function $\widehat{m}_G(y)$, $y \in Y$, is of the form 2.14 (i). We will verify that the characteristic function $\widehat{m}_G(y)$ also satisfies equation (14.7). Since L is a characteristic subgroup of the group Y, if $u, v \in L$, then both sides of equation (14.7) are equal to 1, and if either $u \in L$, $v \notin L$ or $u \notin L$, $v \in L$, then both sides of equation (14.7) are equal to zero. If $u, v \notin L$, then the right-hand side of equation (14.7) is equal to zero. If the left-hand side of equation (14.7) is not equal to zero, then $u+v$, $u+\tilde{\alpha}v$, $u+\tilde{\beta}v \in L$. This implies that $(\tilde{\alpha}-\tilde{\beta})v \in L$. Since $\tilde{\alpha}-\tilde{\beta} \in \mathrm{Aut}(Y)$, we have $v \in L$. The contradiction obtained shows that the left-hand side of equation (14.7) is also equal to zero.

Consider on the group Y the function

$$f(y) = \begin{cases} \widehat{\nu}(y) & \text{if } y \in L, \\ 0 & \text{if } y \notin L. \end{cases}$$

By Proposition 2.12 $f(y)$ is a positive definite function. Hence by the Bochner theorem there exists a distribution $\mu \in \mathrm{M}^1(X)$ such that $\hat{\mu}(y) = f(y)$. Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 3$, be independent identically distributed random variables with values in X and distribution μ. By Lemma 14.11 the characteristic functions $\hat{\mu}_j(y) = \hat{\mu}(y)$, $j = 1, 2, \dots, n$, satisfy equation 10.1 (i) which takes the form (14.7). Then by Lemma 10.1 the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ are independent. Obviously, $\mu \notin I(X)$. In the case when $X = \Delta_2^2$ the lemma is proved.

Let $X = \Delta_p$, where p is a prime number, $p \geq 3$. We follow the scheme of the proof of the lemma for the group $X = \Delta_2^2$. Put $G = X^{(p^2)}$. Then G is a characteristic subgroup of the group X. By Theorem 1.7.1, $Y \cong \mathbb{Z}(p^\infty)$, and by Theorem 1.9.5, $L = A(Y, G) = Y_{(p^2)}$. It is obvious that $Y_{(p^2)} \cong \mathbb{Z}(p^2)$. The subgroup L can be considered as the character group of the group $K = \mathbb{Z}(p^2)$. By Lemma 14.6 there exist independent identically distributed random variables ζ_j, $j = 1, 2, \dots, n$, $n \geq 3$, with values in the group K and distribution $\nu \notin I(K)$ such that the linear forms $L_1 = \zeta_1 + \cdots + \zeta_n$ and $L_2 = \zeta_1 + \cdots + \zeta_{n-2} + 2\zeta_{n-1} - \zeta_n$ are independent. Then by Lemma 10.1 the characteristic functions of the random variables ζ_j satisfy equation 10.1 (i) on the group L which takes the form (14.10).

The verification that the characteristic function $\widehat{m}_G(y)$ also satisfies equation (14.10) and the final part of the proof of the lemma proceed analogously to the case of the group $X = \Delta_2^2$. □

Theorem 14.13. *Let X be a compact totally disconnected group, G be a group of the form* 14.1 (i). *Denote by K one of the groups G, Δ_2, and $\Delta_2 \times G$. Then the following statements hold:*

(I) *Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, 3$, and let ξ_j be independent random variables with values in X and distributions μ_j. Assume that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3$ are independent. If $X = K$, then all μ_j are degenerate distributions. If $X = \mathbb{Z}(3) \times K$, then either all μ_j*

are degenerate distributions or $\mu_{j_1} * E_{x_1} = \mu_{j_2} * E_{x_2} = m_{\mathbb{Z}(3)}$, $x_j \in X$, *at least for two distributions* μ_{j_1} *and* μ_{j_2}. *If* $X = \mathbb{Z}(5) \times K$, *then either all* μ_j *are degenerate distributions or* $\mu_{j_1} * E_{x_1} = m_{\mathbb{Z}(5)}$, $x_1 \in X$, *at least for one distribution* μ_{j_1}.

(II) *If a group* X *is not topologically isomorphic to any of the groups mentioned in* (I), *then for every* $n \geq 3$ *there exist automorphisms* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, *and independent identically distributed random variables* ξ_j *with values in* X *and distribution* $\mu \notin I(X)$ *such that the linear forms* $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ *and* $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ *are independent.*

Proof. (I). In the case when $X = K$ the assertion follows from Lemmas 14.1 and 14.9. If either $X = \mathbb{Z}(3) \times K$ or $X = \mathbb{Z}(5) \times K$, then the proof of the theorem proceeds in the same way as the proof of Theorem 14.7 in case (I). We also use Corollary 14.10 along with Corollary 14.2.

(II). Assume that the group X is not topologically isomorphic to the groups mentioned in (I). If the group X contains a topological direct factor topologically isomorphic to either the group $\Delta_p^{\aleph_0}$, where p is a prime number, $p \geq 3$, or a group of the form 13.24 (ii), then the statement of the theorem follows from Lemmas 13.21, 13.22, and Remark 13.23.

So, we assume that the group X contains no topological direct factor topologically isomorphic to either the group $\Delta_p^{\aleph_0}$, where p is a prime number, $p \geq 3$, or a group of the form 13.24 (ii). By Lemma 13.24 in this case the group X is of the form 13.24 (i). It follows from representation 13.24 (i) that if the group X is not topologically isomorphic to the groups mentioned in (I), then X contains as a topological direct factor a subgroup M such that either $M \cong \Delta_2^2$ or $M \cong \Delta_p$, where p is a prime number, $p \geq 3$, or M is topologically isomorphic to one of the groups mentioned in Lemma 14.6. In this case the statement of the theorem follows from Lemmas 14.12, 14.6, and Remark 13.23. □

Now consider the case when X is a discrete group. Since we assume that all considering groups are second countable, for the discrete groups it means that these groups are countable. We will not specify it.

Lemma 14.14. *Let* X *be a discrete reduced* 2*-primary group such that all its Ulm–Kaplansky invariants are equal to either* 0 *or* 1. *Let* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, $n \geq 2$, *and assume that* ξ_j *are independent random variables with values in* X *and distributions* μ_j. *If the linear forms* $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ *and* $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ *are independent, then all* μ_j *are degenerate distributions.*

Proof. Arguing as in the proof of Theorem 10.3, we reduce the proof of the lemma to the case when $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, where $\delta_j \in \mathrm{Aut}(X)$. By Lemma 10.1 if L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form (10.2). Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (10.2). The lemma will be proved if we show that all $\nu_j \in D(X)$. Therefore we may assume from the beginning that $\hat{\mu}_j(y) \geq 0$, $y \in Y$, $j = 1, 2, \dots, n$.

1. First consider the case when X is either a finite group or an infinite group which is a weak direct product of cyclic groups. Put $X^1 = \bigcap_{n=1}^{\infty} X^{(2^n)}$. Then $X^1 = \{0\}$. It follows from 1.20 that the Ulm sequence of the group X in this case consists of one element $X_0 = X/X^1 \cong X$. We note that the number of cyclic direct factors of order 2^n in the decomposition of the Ulm factor X_0 coincides with the $(n-1)$th Ulm–Kaplansky invariant. Therefore, if X is a finite group, then X is isomorphic to a group of the form 14.1 (i). The assertion of the lemma in this case is proved in Lemma 14.1. If X is an infinite group, then X is isomorphic to a group of the form

$$\mathop{\mathbf{P}^*}_{j=1}^{\infty} \mathbb{Z}(2^{m_j}), \ m_j < m_{j+1}, \quad j = 1, 2, \dots . \tag{14.16}$$

By Theorem 1.7.2,

$$Y = \mathop{\mathbf{P}}_{j=1}^{\infty} A_j, \quad A_j \cong \mathbb{Z}(2^{m_j}).$$

We prove that $\hat{\mu}_j(y) = 1$, $y \in Y$, $j = 1, 2, \dots, n$. Put

$$B_i = \mathop{\mathbf{P}}_{j=i+1}^{\infty} A_j, \quad H_i = Y_{(2)} \cap B_i.$$

It is easy to see that $H_i = Y_{(2)} \cap Y^{(2^{m_{i+1}-1})}$. Since for every natural k the subgroups $Y^{(k)}$ and $Y_{(k)}$ are characteristic, the subgroups H_i are also characteristic. By Lemma 13.9 there exists an open subgroup $B \subset Y$ such that all the characteristic functions $\hat{\mu}_j(y) = 1$, $y \in B$. Taking into account that the subgroups B_i form a base of neighbourhoods of zero, we may assume without loss of generality that $B = B_{k_0}$ for some k_0.

First we check that $\hat{\mu}_j(y) = 1$, $y \in Y_{(2)}$, $j = 1, 2, \dots, n$. Consider the restriction of equation (10.2) to the subgroup $Y_{(2)}$. Since $H_{k_0} \subset B_{k_0}$, all the characteristic functions $\hat{\mu}_j(y) = 1$, $y \in H_{k_0}$. We note that $\varepsilon(H_{i-1} \setminus H_i) = H_{i-1} \setminus H_i$ holds for any automorphism $\varepsilon \in \mathrm{Aut}(Y)$ and any natural i. We conclude from this that if $u, v \in H_{i-1} \setminus H_i$, then $u + \varepsilon v \in H_i$. Substituting $u, v \in H_{k_0-1} \setminus H_{k_0}$ into equation (10.2) we get

$$1 = \mathop{\mathbf{P}}_{j=1}^{n} \hat{\mu}_j(u) \mathop{\mathbf{P}}_{j=1}^{n} \hat{\mu}_j(\tilde{\delta}_j v), \quad u, v \in H_{k_0-1} \setminus H_{k_0}. \tag{14.17}$$

It follows from (14.17) that all characteristic functions $\hat{\mu}_j(y) = 1$, $y \in H_{k_0-1}$. Arguing as above after k_0 steps we obtain that $\hat{\mu}_j(y) = 1$, $y \in H_0 = Y_{(2)}$, $j = 1, 2, \dots, n$.

Since all the characteristic functions $\hat{\mu}_j(y) = 1$, $y \in Y_{(2)}$, by Corollary 2.13 every characteristic function $\hat{\mu}_j(y)$ takes a constant value on each coset $y + Y_{(2)}$ and induces a characteristic function $g_j([y])$ on the factor group $Y/Y_{(2)}$ by the formula $g_j([y]) = \hat{\mu}_j(y)$, $y \in [y]$. On the other hand, since $Y_{(2)}$ is a characteristic subgroup of the group Y, every automorphism $\tilde{\delta}_j$ induces a topological automorphism ε_j on the factor group $Y/Y_{(2)}$ by the formula $\varepsilon_j[y] = [\tilde{\delta}_j y]$, $y \in [y]$. Therefore we may consider equation (10.2) on the factor group $Y/Y_{(2)}$. We have

$$\mathbf{P}_{j=1}^{n} g_j([u] + \varepsilon_j[v]) = \mathbf{P}_{j=1}^{n} g_j([u]) \ \mathbf{P}_{j=1}^{n} g_j(\varepsilon_j[v]), \quad [u], [v] \in Y/Y_{(2)}.$$

The factor group $Y/Y_{(2)}$ is of the same form as the group Y, i.e., $Y/Y_{(2)}$ is a direct product of groups each of which is isomorphic to the group $\mathbb{Z}(2^{l_k})$, and all l_k are different. Hence as has been proved above, $g_j([y]) = 1$, $[y] \in (Y/Y_{(2)})_{(2)}$, $j = 1, 2, \ldots, n$. Returning to the original characteristic functions $\hat{\mu}_j(y)$, we obtain that all the characteristic functions $\hat{\mu}_j(y) = 1$, $y \in Y_{(4)}$. Arguing as above we show that $\hat{\mu}_j(y) = 1$, $y \in Y_{(2^m)}$, $j = 1, 2, \ldots, n$, for all m. Since the subgroups $Y_{(2^{m_{k_0}})}$ and B_{k_0} generate the group Y, by Proposition 2.13, $\hat{\mu}_j(y) = 1$, $y \in Y$, i.e., $\mu_j = E_0$, $j = 1, 2, \ldots, n$.

2. We now consider the general case. Put

$$X^0 = X, \quad X^1 = \bigcap_{n=1}^{\infty} X^{(2^n)}, \quad X^{\sigma+1} = (X^{\sigma})^1, \quad X^{\rho} = \bigcap_{\sigma<\rho} X^{\sigma},$$

if ρ is a limit ordinal number. Let τ be the least ordinal number for which $X^{\tau} = \{0\}$. By Theorem 1.20.1, τ is a countable ordinal number, and each of the groups $X_{\sigma} = X^{\sigma}/X^{\sigma+1}$ is a weak direct product of cyclic groups, moreover the group X_{σ} contains elements of arbitrarily large order if $\sigma + 1 < \tau$. Since the number of cyclic direct factors of order 2^n in the decomposition X_{σ} coincides with the $\boldsymbol{\omega}\sigma + n - 1)$th Ulm–Kaplansky invariant, the Ulm factors X_{σ} are of the form

$$X_{\sigma} \cong \mathbf{P}_{j=1}^{\infty}{}^{*} \, \mathbb{Z}(2^{m_{\sigma,j}}), \quad m_{\sigma,j} < m_{\sigma,j+1} \text{ if } \sigma + 1 < \tau. \tag{14.18}$$

If the ordinal number $\tau - 1$ exists, the group $X_{\tau-1}$ is isomorphic to either a group of the form 14.1 (i) or a group of the form (14.16).

Set $L = A(Y, X^1)$. Obviously, X^1 is a characteristic subgroup of the group X. By Corollary 13.12, L is a characteristic subgroup of the group Y. By Theorem 1.9.1, $X^1 = A(X, L)$, and hence by Theorem 1.9.2, $L \cong (X_0)^*$. Consider the restriction of equation (10.2) to the subgroup L. Since the group X_0 is of the form (14.18), as has been proved in case 1 $\hat{\mu}_j(y) = 1$, $y \in L$, $j = 1, 2, \ldots, n$. By Proposition 2.13, $\sigma(\mu_j) \subset A(X, L) = X^1$. Obviously, X^{κ} is a characteristic subgroup of the group X for all $\kappa < \tau$. Assume that the inclusions $\sigma(\mu_j) \subset X^{\kappa}$, $j = 1, 2, \ldots, n$, $\kappa < \rho$, have already been proved. Two cases are possible: the ordinal number $\rho - 1$ exists and ρ is a limit ordinal number. In the first case we argue as above, replacing X by $X^{\rho-1}$. Then we prove that all $\sigma(\mu_j) \subset X^{\rho}$. In the second case $X^{\rho} = \bigcap_{\delta<\rho} X^{\delta}$, and then also all $\sigma(\mu_j) \subset X^{\rho}$. Hence $\sigma(\mu_j) \subset X^{\tau} = \{0\}$, i.e., $\mu_j = E_0$, $j = 1, 2, \ldots, n$. □

Lemma 14.15. *Let either $X = \mathbb{Z}(2^{\infty})$ or $X = \mathbb{Z}(2^{\infty}) \times G$, where G is a discrete reduced 2-primary group such that all its Ulm–Kaplansky invariants are equal either* 0 *or* 1. *Let* $\alpha_j,\ \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, $n \geq 2$, *and let* ξ_j *be independent random variables with values in* X *and distributions* μ_j. *If the linear forms* $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ *and* $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ *are independent, then all* μ_j *are degenerate distributions.*

Proof. By Lemma 10.1 it follows from the independence of the linear forms L_1 and L_2 that the characteristic functions $\hat\mu_j(y)$ satisfy equation 10.1 (i). Put $\nu_j = \mu_j * \bar\mu_j$. By 2.7 (c) and 2.7 (d), $\hat\nu_j(y) = |\hat\mu_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat\nu_j(y)$ also satisfy equation 10.1 (i). The lemma will be proved if we show that all $\nu_j \in D(X)$. Therefore we can assume from the beginning that $\hat\mu_j(y) \geq 0$, $y \in Y$.

1. Let $X = \mathbb{Z}(2^\infty)$, then $Y \cong \Delta_2$. Put $Y_i = Y^{(2^i)}$. It follows from the definition of the topology in the group Δ_2 that all sets Y_i form a base of neighbourhoods of zero. By Lemma 13.9 there exists an open subgroup B of the group Y such that $\hat\mu_j(y) = 1$, $y \in B$, $j = 1, 2, \dots, n$. We may assume without loss of generality that $B = Y_{k_0}$ for some k_0. By Proposition 2.13, $\sigma(\mu_j) \subset A(X, Y_{k_0}) = X_{(2^{k_0})} \cong \mathbb{Z}(2^{k_0})$. Since $X_{(2^{k_0})}$ is a characteristic subgroup of the group X, the statement of the lemma follows from Lemma 14.1.

2. Let $X = \mathbb{Z}(2^\infty) \times G$, where G is a discrete reduced 2-primary group such that all its Ulm–Kaplansky invariants are equal to either 0 or 1. Set $H = A(Y, \mathbb{Z}(2^\infty))$. By Theorem 1.9.1, $A(X, H) = \mathbb{Z}(2^\infty)$. Since $Z(2^\infty)$ is the maximal divisible subgroup of the group X, $Z(2^\infty)$ is a characteristic subgroup of X. By Corollary 13.12, H is a characteristic subgroup of the group Y. Consider the restriction of equation 10.1 (i) to the subgroup H. Since $H \cong G^*$, we conclude from Lemma 14.14 that $\hat\mu_j(y) = 1$, $y \in H$, $j = 1, 2, \dots, n$. Hence by Proposition 2.13 all $\sigma(\mu_j) \subset A(X, H) = \mathbb{Z}(2^\infty)$. Now the assertion of the lemma follows from case 1. □

Lemmas 14.14 and 14.15 yield directly

Corollary 14.16. *Let* H *be the character group of a discrete reduced* 2*-primary group such that all its Ulm–Kaplansky invariants are equal to either* 0 *or* 1. *Let either* $Y = H$ *or* $Y = \Delta_2$ *or* $Y = \Delta_2 \times H$. *Let* $\tilde\alpha_j, \tilde\beta_j \in \mathrm{Aut}(Y)$, $j = 1, 2, \dots, n$, $n \geq 2$, *and let* $\hat\mu_j(y)$ *be characteristic functions on the group* Y *satisfying equation 10.1* (i). *Then* $\hat\mu_j(y)$ *are of the form*

$$\hat\mu_j(y) = (x_j, y), \quad x_j \in X,\ j = 1, 2, \dots, n.$$

Lemma 14.17. *Let* X *be a discrete reduced* 2*-primary group such that not all its Ulm–Kaplansky invariants are equal to either* 0 *or* 1. *Then the group* X *contains a subgroup* M, $M \cong (\mathbb{Z}(2^k))^2$, *such that any automorphism of* M *extends to an automorphism of the group* X.

Proof. Let A be an arbitrary p-primary group. As is well known (see ([50], § 33) there exists a subgroup B of A such that B has the following properties:

(a) B is a weak direct product of cyclic groups;

(b) B is pure in A;

(c) A/B is divisible.

Such a subgroup B is called a *basic subgroup of* A. Denote by B_n a weak direct product of cyclic factors in B of order p^n, i.e., $B_n \cong \mathbf{P}^* \mathbb{Z}(p^n)$ and

$$B = \mathop{\mathbf{P}^*}_{n=1}^{\infty} B_n.$$

Then the decomposition holds

$$A = B_1 \times \cdots \times B_n \times C_n, \quad n = 1, 2, \ldots, \tag{14.19}$$

where C_n is the subgroup of A generated by $\mathbf{P}^*_{i \geq n+1} B_i$ and $A^{(p^n)}$ ([50], Theorem 32.4).

Pass now to the proof of the lemma. By Theorem 1.20 every Ulm factor of the group X is a weak direct product of cyclic 2-primary groups. Therefore it follows from the condition of the lemma that there exists an Ulm factor X_σ such that X_σ has a direct factor L which is isomorphic to the group $(\mathbb{Z}(2^k))^2$ for some natural k. We have $X_\sigma = X^\sigma / X^{\sigma+1}$. Let B be a basic subgroup of the group X^σ. Then $X^{\sigma+1} \cap B = \{0\}$. Denote by B_0 the image of the subgroup B under the natural homomorphism $\pi : X^\sigma \mapsto X_\sigma$. Then B_0 is a basic subgroup of the group X_σ and π is an isomorphism between B and B_0 ([50], § 34). Since X_σ is a weak direct product of cyclic groups, a basic subgroup of X_σ is itself. Taking into account that all basic subgroups of a p-primary group are isomorphic ([50], Theorem 35.2), we have $X_\sigma \cong B_0$. This implies that $X_\sigma \cong B$. Hence the subgroup B has a direct factor M isomorphic to $(\mathbb{Z}(2^k))^2$. We have $B = M \times N$, where N is a weak direct product of cyclic groups, because N is a subgroup of B. We get a decomposition of the group B into a weak direct product of cyclic groups. Let B_n be a weak direct product of cyclic factors in B of order 2^n. Then M is a direct factor of B_k. Applying decomposition (14.19) to the group X^σ we conclude that M is a direct factor of X^σ. Hence every automorphism of the subgroup M extends to an automorphism of the group X^σ. By the Zippin theorem every automorphism of the subgroup X^σ extends to an automorphism of the group X ([51], Theorem 77.4). Thus every automorphism of the subgroup M extends to an automorphism of the group X. □

Lemma 14.18. *For each of the groups $X = (\mathbb{Z}(2^\infty))^2$ and $X = \mathbb{Z}(p^\infty)$, where p is a prime number, $p \geq 3$, and for every $n \geq 3$ there exist automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ and independent identically distributed random variables ξ_j, $j = 1, 2, \ldots, n$, with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ are independent.*

Proof. Let $X = (\mathbb{Z}(2^\infty))^2$. Denote by $(a, b), a, b \in \mathbb{Z}(2^\infty)$, elements of the group X. Let automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ be of the form

$$\alpha(a, b) = (a, a + b), \quad \beta(a, b) = (b, a), \quad a, b \in \mathbb{Z}(2^\infty).$$

Consider the subgroup $K = X_{(2)}$. Obviously, $K \cong (\mathbb{Z}(2))^2$. The subgroup K is characteristic and the restrictions of the automorphisms α and β to K coincide with the automorphisms defined by (14.5). By Lemma 14.6 there exist independent identically distributed random variables ξ_j, $j = 1, 2, \ldots, n$, $n \geq 3$, with values in the group K and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-2} + \alpha\xi_{n-1} + \beta\xi_n$ are independent. The statement of the lemma follows from Remark 13.23.

Let $X = \mathbb{Z}(p^\infty)$, where p is a prime number, $p \geq 3$. The scheme of the proof of the lemma in this case is the same as for the group $X = (\mathbb{Z}(2^\infty))^2$. Set $K = X_{(p^2)}$. It is obvious that $K \cong \mathbb{Z}(p^2)$, and the subgroup K is characteristic. By Lemma 14.6 there exist independent identically distributed random variables ξ_j, $j = 1, 2, \ldots, n$, $n \geq 3$, with values in the group K and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_{n-2} + 2\xi_{n-1} - \zeta_n$ are independent. Obviously, the automorphisms f_2 and $-I$ extend from K to automorphisms of the group X. The statement of the lemma also follows from Remark 13.23. □

Theorem 14.19. *Let X be a discrete torsion group, G be a discrete reduced 2-primary group such that all its Ulm–Kaplansky invariants are equal to either 0 or 1. Denote by K one of the groups G, $\mathbb{Z}(2^\infty)$, and $\mathbb{Z}(2^\infty) \times G$. Then the following statements hold:*

(I) *Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, 3$, and assume that ξ_j are independent random variables with values in X and distributions μ_j. Assume that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3$ are independent. If $X = K$, then all μ_j are degenerate distributions. If $X = \mathbb{Z}(3) \times K$, then either all μ_j are degenerate distributions or $\mu_{j_1} * E_{x_1} = \mu_{j_2} * E_{x_2} = m_{\mathbb{Z}(3)}$, $x_j \in X$, at least for two distributions μ_{j_1} and μ_{j_2}. If $X = \mathbb{Z}(5) \times K$, then either all μ_j are degenerate distributions or $\mu_{j_1} * E_{x_1} = m_{\mathbb{Z}(5)}$, $x_1 \in X$, at least for one distribution μ_{j_1}.*

(II) *If a group X is not isomorphic to any of the groups mentioned in (I), then for every $n \geq 3$ there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \ldots, n$, and independent identically distributed random variables ξ_j with values in X and distribution $\mu \notin I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent.*

Proof. (I). If $X = K$, the statement of the theorem follows from Lemmas 14.14 and 14.15. If either $X = \mathbb{Z}(3) \times K$ or $X = \mathbb{Z}(5) \times K$, then the proof of the theorem is similar to the proof of case (I) of Theorem 14.7.

(II). Assume that the group X is not topologically isomorphic to the groups mentioned in (I). By Theorem 1.19.1, X is a weak direct product of its p-components X_p. By Theorem 1.19.2 each p-primary subgroup X_p can be represented as a direct product $X_p = D_p \times N_p$, where D_p is the maximal divisible subgroup of X_p and N_p is a countable reduced p-primary group. By Theorem 1.19.3 the group D_p is represented in its turn as a weak direct product of groups each of which is isomorphic to the group $Z(p^\infty)$. We use the fact that every reduced p-primary group has a cyclic direct factor ([50], § 27). Hence the group N_p has a direct factor $M \cong \mathbb{Z}(p^k)$ for some natural k. We conclude from this that if a group X is not isomorphic to any of the groups mentioned in (I), then the group X has a direct factor isomorphic to either $(\mathbb{Z}(2^\infty))^2$

or $\mathbb{Z}(p^\infty)$, where p is a prime number, $p \geq 3$, or a reduced 2-prime group such that not all its Ulm–Kaplansky invariants are equal to either 0 or 1, or one of the groups mentioned in Lemma 14.6. The statement of the theorem follows from Lemmas 14.18, 14.17, 14.6 and Remark 13.23. $\square$

Now we discuss the case when X is either a compact group or a discrete torsion group and the number of independent random variables $n \geq 4$. We need the following.

Lemma 14.20. *Assume that a group X contains a compact subgroup F with the following properties:*

(a) $F \ncong Z(3)$;

(b) *there exists an automorphism $\delta \in \mathrm{Aut}(F)$ that extends to a topological automorphism of the group X;*

(c) *$I - \delta$ is an epimorphism of the subgroup F.*

Then for every $n \geq 4$ there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, and independent random variables ξ_j with values in X and distributions $\mu_j \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ are independent.

Proof. In view of Remark 13.23 in the proof of the lemma we can assume from the beginning that $X = F$. Then it follows from (a) and (c) that $Y \ncong \mathbb{Z}(3)$ and $Y \ncong \mathbb{Z}(2)$. Hence we can take nonzero elements $y_1, y_2 \in Y$ such that $\{y_1, -y_1\} \cap \{y_2, -y_2\} = \emptyset$. Consider on the group X the functions

$$\rho_j(x) = 1 + (1/2)\,\mathrm{Re}(x, y_j).$$

Then $\rho_j(x) > 0,\ x \in X$, and

$$\int\limits_X \rho_j(x) d m_X(x) = 1, \quad j = 1, 2.$$

Denote by λ_j the distribution on the group X with density $\rho_j(x)$ with respect to the Haar distribution m_X. Let $a(y_j) = \frac{1}{4}$ if $2y_j \neq 0$ and $a(y_j) = \frac{1}{2}$ if $2y_j = 0$. It is easily seen that the characteristic functions of the distributions λ_j have the form:

$$\hat{\lambda}_j(y) = \begin{cases} 1 & \text{if } y = 0, \\ a(y_j) & \text{if } y = \pm y_j, \\ 0 & \text{if } y \notin \{0, \pm y_j\}. \end{cases} \tag{14.20}$$

Let ξ_j be independent random variables with values in X and distributions μ_j, where $\mu_1 = \mu_3 = \lambda_1$, $\mu_2 = \mu_4 = \lambda_2$, and μ_j, $j \geq 5$ are arbitrary distributions on X such that $\mu_j \notin \Gamma(X) * I(X)$. Assume that $\delta_1 = \delta_2 = I$, $\delta_3 = \delta_4 = \delta$, and $\delta_j \in \mathrm{Aut}(X)$ are arbitrary automorphisms. It is obvious that all $\mu_j \notin \Gamma(X) * I(X)$.

We will show that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form

$$\begin{aligned}\hat{\lambda}_1(u+v)\hat{\lambda}_2(u+v)\hat{\lambda}_1(u+\tilde{\delta}v)\hat{\lambda}_2(u+\tilde{\delta}v)\prod_{j=5}^{n}\hat{\mu}_j(u+\tilde{\delta}_j v)\\ =\hat{\lambda}_1^2(u)\hat{\lambda}_1(v)\hat{\lambda}_1(\tilde{\delta}v)\hat{\lambda}_2^2(u)\hat{\lambda}_2(v)\hat{\lambda}_2(\tilde{\delta}v)\prod_{j=5}^{n}\hat{\mu}_j(u)\prod_{j=5}^{n}\hat{\mu}_j(\tilde{\delta}_j v),\end{aligned} \tag{14.21}$$

$u, v \in Y$. Then by Lemma 10.1 it follows from this that the linear forms $L_1 = \xi_1 + \cdots + \xi_n$ and $L_1 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$ are independent, and the lemma will be proved.

Obviously, equation (14.21) is true if either $u = 0$ or $v = 0$. We will show that equation (14.21) holds for all $u \neq 0$ and $v \neq 0$. So, assume that $u \neq 0$ and $v \neq 0$. Then we conclude from (14.20) that the right-hand side of equation (14.21) is equal to zero. We will verify that the left-hand side of equation (14.21) is also equal to zero. Assume that $\hat{\lambda}_1(u+v)\hat{\lambda}_2(u+v)\hat{\lambda}_1(u+\tilde{\delta}v)\hat{\lambda}_2(u+\tilde{\delta}v) \neq 0$. This implies that $u+v, u+\tilde{\delta}v \in \{0, y_1, -y_1\} \cap \{0, y_2, -y_2\} = \{0\}$. Hence

$$\begin{cases} u+v=0, \\ u+\tilde{\delta}v = 0. \end{cases} \tag{14.22}$$

It follows from (14.22) that $(I-\tilde{\delta})v = 0$. Since $I-\delta$ is an epimorphism of the group X, by 1.13 (b), $I-\tilde{\delta}$ is a monomorphism of the group Y. Hence the equality $(I-\tilde{\delta})v = 0$ implies that $v = 0$, contrary to the assumption. Thus the left-hand side of equation (14.21) is also equal to zero for $u \neq 0$ and $v \neq 0$. So, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (14.21). □

Proposition 14.21. *Let X be a compact group, G be a group of the form* 14.1 (i). *Denote by K one of the groups G, Δ_2, and $\Delta_2 \times G$. If the group X is not topologically isomorphic to either K or $\mathbb{Z}(3) \times K$, then for every $n \geq 4$ there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, and independent random variables ξ_j with values in X and distributions $\mu_j \notin \Gamma(X) * I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent.*

Proof. Consider the connected component of zero c_X of the group X and assume that $c_X \neq \{0\}$. By Theorem 1.9.6, $(c_X)^{(2)} = c_X$. Hence the subgroup $F = c_X$ and the automorphism $-I$ satisfy the conditions of Lemma 14.20. In this case the proposition follows from Lemma 14.20.

If $c_X = \{0\}$, then X is a totally disconnected group. We apply Theorem 14.13 and conclude that the proposition holds for all groups X except $X = \mathbb{Z}(5) \times K$. Let $X = \mathbb{Z}(5) \times K$. Notice that the subgroup $F = \mathbb{Z}(5)$ and the automorphism $-I$ satisfy the conditions of Lemma 14.20, and thus the proposition follows. □

Proposition 14.22. *Let X be a discrete torsion group, G be a discrete reduced 2-primary group such that all its Ulm–Kaplansky invariants are equal to either 0 or 1. Denote by K one of the groups G, $\mathbb{Z}(2^\infty)$, and $\mathbb{Z}(2^\infty) \times G$.*

If the group X is not isomorphic to either K or $\mathbb{Z}(3) \times K$, then for every $n \geq 4$ there exist automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, and independent random variables ξ_j with values in X and distributions $\mu_j \notin I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent.

Proof. We apply Theorem 14.19. It follows from Theorem 14.19 that the proposition holds for all groups X except $X = \mathbb{Z}(5) \times K$. Let $X = \mathbb{Z}(5) \times K$. Notice that the subgroup $F = \mathbb{Z}(5)$ and the automorphism $-I$ satisfy the conditions of Lemma 14.20, and thus the proposition follows. □

At the end of this section we will prove the following statement.

Theorem 14.23. *Let*

$$\text{(i)} \quad X = \mathbb{R}^m \times \Delta_2 \times \mathbb{Z}(2^\infty) \times D,$$

where D is a discrete group such that its torsion part b_D is a reduced 2-primary group such that all its Ulm–Kaplansky invariants are equal to either 0 or 1. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2, \dots, n$, $n \geq 2$, and let ξ_j be independent random variables with values in X and distributions μ_j. If the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, then all $\mu_j \in \Gamma(X)$.

Proof. By Lemma 10.1, if the linear forms L_1 and L_2 are independent, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i). Put $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$, $j = 1, 2, \dots, n$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation 10.1 (i). The group X has no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Hence if we prove that $\nu_j \in \Gamma(X)$, then applying Theorem 4.6, we get that $\mu_j \in \Gamma(X)$. Therefore we can assume from the beginning that the characteristic functions $\hat{\mu}_j(y) \geq 0$.

It is easy to see that $b_X = \Delta_2 \times \mathbb{Z}(2^\infty) \times b_D$ and $\mathbb{R}^m \times b_X$ is a characteristic subgroup. Taking into account Proposition 13.8 we can assume that D is a torsion group.

Obviously that $Y \cong \mathbb{R}^m \times \mathbb{Z}(2^\infty) \times \Delta_2 \times L$, where $L = D^*$. We will assume that $Y = \mathbb{R}^m \times \mathbb{Z}(2^\infty) \times \Delta_2 \times L$. Note that the subgroup $\mathbb{Z}(2^\infty) \times D$ of X is closed and characteristic, because it consists of all elements of finite order of the group X. By Corollary 13.12 its annihilator $A(Y, \mathbb{Z}(2^\infty) \times D) = \mathbb{R}^m \times \mathbb{Z}(2^\infty)$ is a characteristic subgroup of the group Y. Hence $\mathbb{Z}(2^\infty)$ is also a characteristic subgroup of the group Y. Consider the restriction of equation 10.1 (i) to the subgroup $\mathbb{Z}(2^\infty)$. It follows from Corollary 14.10 that $\hat{\mu}_j(y) = 1$, $y \in \mathbb{Z}(2^\infty)$. Thus by Proposition 2.13 all $\sigma(\mu_j) \subset A(X, \mathbb{Z}(2^\infty)) = K$, where

$$K = \mathbb{R}^m \times \mathbb{Z}(2^\infty) \times b_D. \tag{14.23}$$

Since $\mathbb{Z}(2^\infty)$ is a characteristic subgroup of the group Y, by Corollary 13.12 K is a characteristic subgroup of the group X. Therefore we can prove the theorem assuming that the group X is of the form (14.23). Hence $Y \cong \mathbb{R}^m \times \Delta_2 \times L$. We will assume that $Y = \mathbb{R}^m \times \Delta_2 \times L$. The subgroup $b_Y = \Delta_2 \times L$ is characteristic. Consider the restriction of equation 10.1 (i) to the subgroup $\Delta_2 \times L$. By Corollary 14.16 all characteristic functions $\hat{\mu}_j(y) = 1$, $y \in \Delta_2 \times L$. Thus by Proposition 2.13 we have the inclusions $\sigma(\mu_j) \subset A(X, \Delta_2 \times L) = \mathbb{R}^m$, $j = 1, 2, \dots, n$. Since $\mathbb{R}^m = c_X$, $\mathbb{R}^m$ is a characteristic subgroup of the group X. The proof of the theorem is reduced to the proof for the group $X = \mathbb{R}^m$. Hence the theorem follows from the Ghurye–Olkin theorem (see Remark 13.7). □

15 Random variables with values in the *a*-adic solenoid $\Sigma_{\boldsymbol{a}}$

Let X be a compact connected group. According to Theorem 13.26 there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ and automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent. The aim of this section is to study the following problem: Let ξ_1 and ξ_2 be independent random variables with values in the $\boldsymbol{a}$-adic solenoid $X = \Sigma_{\boldsymbol{a}}$ and distributions μ_1 and μ_2. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ and assume that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent. What can one say about the distributions μ_1 and μ_2? It turns out that the answer depends on both the form of the $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ and the topological automorphisms α_j, β_j.

We need some lemmas.

Lemma 15.1. *Let Y be an arbitrary Abelian group, let $q \in \mathbb{Z}$, $q \neq 0$, and let $a(y)$ and $b(y)$ be functions on Y satisfying the equation*

$$\text{(i)} \quad a(u+v)b(u+qv) = a(u)b(u)a(v)b(qv), \quad u, v \in Y,$$

and the conditions

$$\text{(ii)} \quad a(-y) = a(y), \quad b(-y) = b(y), \quad y \in Y.$$

Set $s = 1 - q$. If for a certain $y_0 \in Y^{(s)}$ the inequality $a(y_0)b(y_0) \neq 0$ holds, then there exists a subgroup $M \subset Y$ such that $a(y)b(y) \neq 0$, $y \in M$.

Proof. Putting $u = -qy$, $v = y$ and then $u = y$, $v = -y$ in 15.1 (i) and taking into account 15.1 (ii) we get

$$a(sy) = a(qy)b^2(qy)a(y), \quad y \in Y, \tag{15.1}$$

$$b(sy) = a^2(y)b(y)b(qy), \quad y \in Y. \tag{15.2}$$

By the condition of the lemma $y_0 = sz_0$, $z_0 \in Y$. Substituting z_0 for y into (15.1) and (15.2) we conclude that

$$a(z_0) \neq 0, \; a(qz_0) \neq 0, \quad b(z_0) \neq 0, \; b(qz_0) \neq 0. \tag{15.3}$$

Putting $u = z_0$, $v = kz_0$ and then $u = qz_0$, $v = kz_0$, $k \in \mathbb{Z}$, in equation 15.1 (i) we obtain

$$a((k+1)z_0))b((kq+1)z_0) = a(z_0))b(z_0)a(kz_0)b(kqz_0), \tag{15.4}$$
$$a((q+k)z_0))b((k+1)qz_0) = a(qz_0))b(qz_0)a(kz_0)b(kqz_0). \tag{15.5}$$

Taking into account (15.3), it follows by induction from (15.4), (15.5) that $a(kz_0) \neq 0$, $b(kqz_0) \neq 0$, $k \in \mathbb{Z}$. The subgroup $M = \{kqz_0\}_{k\in\mathbb{Z}}$ is the required one. □

Lemma 15.2. *Let M be an arbitrary Abelian group, $a(y)$ and $b(y)$ be functions on M satisfying equation* 15.1 (i)*, conditions* 15.1 (ii)*, and the conditions*

$$\text{(i)} \quad 0 < a(y) \leq 1, \quad 0 < b(y) \leq 1, \quad a(0) = b(0) = 1.$$

Put $s = 1 - q$. Then on the subgroup $M^{(sq)}$ the following representation holds:

$$\text{(ii)} \quad a(y) = \exp\{-\varphi_1(y)\}, \quad b(y) = \exp\{-\varphi_2(y)\},$$

where $\varphi_j(y) \geq 0$ and each of the functions $\varphi_j(y)$ satisfies equation 2.16 (ii).

Proof. Set $\varphi_1(y) = -\ln a(y)$, $\varphi_2(y) = -\ln b(y)$. It follows from 15.1 (i) that

$$\varphi_1(u+v) + \varphi_2(u+qv) = A(u) + B(v), \quad u, v \in M, \tag{15.6}$$

where $A(u) = \varphi_1(u) + \varphi_2(u)$, $B(v) = \varphi_1(v) + \varphi_2(qv)$.

We use the finite difference method to solve equation (15.6). Let k be an arbitrary element of Y. Substitute $u - qk$ for u and $v + k$ for v in equation (15.6) and subtract equation (15.6) from the resulting equation. We get

$$\varphi_1(u+v+sk) - \varphi_1(u+v) = \Delta_{-qk}A(u) + \Delta_k B(v), \quad u, v, k \in M. \tag{15.7}$$

Putting $v = 0$ in (15.7) and subtracting the resulting equation from (15.7) we obtain

$$\begin{aligned} \varphi_1(u+v+sk) - \varphi_1(u+v) - \varphi_1(u+sk) + \varphi_1(u) \\ = \Delta_k B(v) - \Delta_k B(0), \quad u, v, k \in M. \end{aligned} \tag{15.8}$$

Put $v = sk$ in (15.8). We get

$$\Delta_{sk}^2 \varphi_1(u) = d_1(k), \quad u, k \in M, \tag{15.9}$$

where $d_1(y)$ is a function on M. Applying the operator Δ_{sk} to both sides of (15.9) we find

$$\Delta_{sk}^3 \varphi_1(u) = 0, \quad u, k \in M. \tag{15.10}$$

We conclude from (15.10) that the function $\varphi_1(y)$ satisfies the equation

$$\Delta_l^3 \varphi_1(u) = 0, \quad u, l \in M^{(s)}. \tag{15.11}$$

Taking into account (i) we get $\varphi_1(y) \ge 0$. By Lemma 10.10 it follows from (15.11) that the function $\varphi_1(y)$ satisfies equation 2.16 (ii). The desired representation for the function $\varphi_1(y)$ is proved.

For the function $\varphi_2(y)$ we argue as above. Let k be an arbitrary element of Y. Substitute $u-k$ for u and $v+k$ for v in equation (15.6) and subtract equation (15.6) from the resulting equation. We get

$$\varphi_2(u+qv-sk)-\varphi_2(u+qv)=\Delta_{-k}A(u)+\Delta_k B(v), \quad u,v,k\in M. \tag{15.12}$$

Putting $v=0$ in (15.12) and subtracting the resulting equation from (15.12) we obtain

$$\begin{aligned}\varphi_2(u+qv-sk)-\varphi_2(u+qv)-\varphi_2(u-sk)+\varphi_2(u)\\ =\Delta_k B(v)-\Delta_k B(0), \quad u,v,k\in M.\end{aligned} \tag{15.13}$$

Substitute $k=qt$, $v=-st$, $t\in M$, in (15.13). We get

$$\Delta^2_{-sqt}\varphi_2(u)=d_2(t), \quad u,t\in M, \tag{15.14}$$

where $d_2(y)$ is a function on M. Applying the operator Δ_{-sqt} to both sides of (15.14) we find

$$\Delta^3_{-sqt}\varphi_2(u)=0, \quad u,t\in M. \tag{15.15}$$

We conclude from (15.15) that the function $\varphi_2(y)$ satisfies the equation

$$\Delta_l^3\varphi_2(u)=0, \quad u,l\in M^{(sq)}. \tag{15.16}$$

Taking into account (i) we get $\varphi_2(y)\ge 0$. By Lemma 10.10 it follows from (15.16) that the function $\varphi_2(y)$ satisfies equation 2.16 (ii). The desired representation for the function $\varphi_2(y)$ is also proved. □

Corollary 15.3. *Assume that the conditions of Lemma 15.2 are satisfied, and let M be a subgroup of $\mathbb{Q}$. Then the following representations hold:*

$$\text{(i)} \quad a(y)=e^{-\sigma_1 y^2}, \quad b(y)=e^{-\sigma_2 y^2}, \quad \sigma_1\ge 0,\ \sigma_2\ge 0,\ y\in M^{(sq)}.$$

Lemma 15.4. *Let $X=\Sigma_{\boldsymbol{a}}$, and let K be a compact subgroup of X such that its annihilator $A(Y,K)\ncong\mathbb{Z}$. Let $\mu\in \mathrm{M}^1(X)$ and $\nu=\mu*\bar\mu$. If $\nu=\gamma*m_K$, where $\gamma\in\Gamma(X)$, then $\mu\in\Gamma(X)*I(X)$.*

Proof. Let $p\colon X\mapsto X/K$ be the natural homomorphism. Put $L=A(Y,K)$ and use representation 2.14 (i) for the characteristic function $\widehat{m}_K(y)$. Since by Theorem 1.9.2, $(X/K)^*\cong L$, it follows from Corollary 2.11 that the restriction of the characteristic function $\hat\nu(y)$ to L is the characteristic function of the Gaussian distribution $p(\nu)$ on the factor group X/K. We conclude from $L\ncong\mathbb{Z}$ that $X/K\ncong\mathbb{T}$. It is clear that the factor group X/K contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. We have $p(\nu)=p(\mu)*p(\bar\mu)$. Applying Theorem 4.6 to the factor group X/K we get

$p(\mu) \in \Gamma(X/K)$. Hence the restriction of the characteristic function $\hat{\mu}(y)$ to L is the characteristic function of a Gaussian distribution. Thus we have the representation

$$\hat{\mu}(y) = \begin{cases} ([x_0], y)\exp\{-\varphi(y)\} & \text{if } y \in L, \\ 0 & \text{if } y \notin L, \end{cases} \tag{15.17}$$

where $[x_0] \in X/K$, and $\varphi(y) = \sigma y^2$, $\sigma \geq 0$. The function $\varphi(y)$ extends in the natural way from L to Y. Denote by $\tilde{\varphi}(y)$ the extended function. Take $x_0 \in [x_0]$ and consider a Gaussian distribution λ on the group X with the characteristic function

$$\hat{\lambda}(y) = (x_0, y)\exp\{-\tilde{\varphi}(y)\}, \quad y \in Y. \tag{15.18}$$

We conclude from (15.18) and representation 2.14 (i) for the characteristic function $\widehat{m}_K(y)$ that $\hat{\mu}(y) = \hat{\lambda}(y)\widehat{m}_K(y)$. In view of 2.7 (b) and 2.7 (c) we get $\mu = \lambda * m_K$. □

Lemma 15.5. *Let Y be a subgroup of $\mathbb{Q}$, and let F be a subgroup of Y. Consider $\mathbb{Q}$ as a subgroup of $\mathbb{R}$ and assume that the subgroup F is dense in $\mathbb{R}$ in the topology of $\mathbb{R}$, i.e., in the topology induced on F by the standard topology of $\mathbb{R}$. Let $g(y)$ be a positive definite function on Y such that its restriction to F is a continuous function in the topology of $\mathbb{R}$. Then the following statements hold:*

(a) *For any $u \in Y$ the restriction of the function $g(y)$ to the coset $u + F$ is a uniformly continuous function in the topology of $\mathbb{R}$;*

(b) *for any $u \in Y$ there exists the limit*

$$g_u = \lim_{y \to 0, y \in u+F} g(y);$$

(c) *assume that $g_u = 1$ for some $u \in Y$ and denote by V the subgroup of Y generated by u and F. Then $g(y)$ is a uniformly continuous function on V in the topology of $\mathbb{R}$.*

Proof. Note that by the Bochner theorem inequality 2.7 (g) holds for any normalized positive definite function $g(y)$. Statement (a) follows from this. Statement (b) follows directly from (a). To prove (c) we notice that inequality (2.3) is true for any normalized positive definite function $g(y)$. We conclude from (2.3) that if $g_u = 1$, then $g_v = 1$ for any $v \in V$. Since the factor group V/F is finite, inequality 2.7 (g) implies that the restriction of the function $g(y)$ to V is uniformly continuous. □

15.6 Notation. Let ξ_1 and ξ_2 be independent random variables with values in the group $X = \Sigma_{\boldsymbol{a}}$ and distributions μ_1 and μ_2. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. Consider the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ and assume that L_1 and L_2 are independent. Taking into consideration new independent random variables $\zeta_j = \alpha_j\xi_j$ we reduce the study of the problem to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_j \in \mathrm{Aut}(X)$. According to 1.14 (d), any topological automorphism δ of the group X is of the form

$$\delta = f_p f_q^{-1},$$

for some relatively prime p and q, where f_p, $f_q \in \mathrm{Aut}(X)$. Note that for any $\delta \in \mathrm{Aut}(X)$ the linear forms L_1 and L_2 are independent if and only if the linear forms L_1 and δL_2 are independent. Hence without loss of generality, we can assume from the beginning that $L_1 = \xi_1 + \xi_2$, $L_2 = p\xi_1 + q\xi_2$, where $p, q \in \mathbb{Z}$, $pq \neq 0$, $pq \neq 1$ and p and q are relatively prime.

Put $s = p - q$, and decompose $|s|$ into prime factors $|s| = s_1^{k_1} \dots s_l^{k_l}$. Consider the corresponding homomorphisms f_{s_j}, $j = 1, 2, \dots, l$. Let $f_{s_{j_1}}, \dots, f_{s_{j_r}}$ be the homomorphisms which are topological automorphisms of X. Denote by H the subgroup of Y of the form

$$H = \left\{ \frac{m}{s_{j_1}^{n_1} \dots s_{j_r}^{n_r}} : m, n_j \in \mathbb{Z} \right\}$$

(in the case when if either $|s| = 1$ or all $f_{s_j} \notin \mathrm{Aut}(X)$, we suppose $H = \mathbb{Z}$). Set $G = H^*$. We fix the notation: f_{s_j}, H, and G throughout this section.

Now we formulate the main result of this section.

Theorem 15.7. *Let $X = \Sigma_{\boldsymbol{a}}$, and let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Assume that $f_p, f_q \in \mathrm{Aut}(X)$, $pq \neq 1$, and p and q are relatively prime. Put $s = p - q$. Decompose $|s|$ into prime factors $|s| = s_1^{k_1} \dots s_l^{k_l}$ and consider the corresponding homomorphisms f_{s_j}, $j = 1, 2, \dots, l$. If the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = p\xi_1 + q\xi_2$ are independent, then the possible distributions μ_1 and μ_2 are described in Tables 1 and 2.*

Table 1

<table>
<tr><td rowspan="2">pq is a composite number</td><td colspan="2">$pq = -1$</td></tr>
<tr><td>$f_2 \in \mathrm{Aut}(X)$</td><td>$f_2 \notin \mathrm{Aut}(X)$</td></tr>
<tr><td>there exist $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$</td><td>$\mu_1, \mu_2 \in \Gamma(X) * I(X)$</td><td>there exist $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$</td></tr>
</table>

Table 2

<table>
<tr><td colspan="4">pq is a prime number</td></tr>
<tr><td rowspan="2">$f_{s_j} \notin \mathrm{Aut}(X)$ at least for one $s_j \neq 2$</td><td colspan="2">$f_{s_j} \in \mathrm{Aut}(X)$ for all $s_j \neq 2$ and $f_2 \notin \mathrm{Aut}(X)$</td><td rowspan="2">$f_s \in \mathrm{Aut}(X)$</td></tr>
<tr><td>s is divisible by 4</td><td>s is not divisible by 4</td></tr>
<tr><td>there exist $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$</td><td>there exist $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$</td><td>at least one of $\mu_j \in \Gamma(X) * I(X)$</td><td>at least one of $\mu_j \in \Gamma(X) * I(X)$</td></tr>
</table>

Proof. By Lemma 10.1 the independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 10.1 (i) which takes the form

$$\hat{\mu}_1(u + pv)\hat{\mu}_2(u + qv) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(pv)\hat{\mu}_2(qv), \quad u, v \in Y. \tag{15.19}$$

We will study the solutions of this equation. Three cases are possible: pq is a composite number, $pq = -1$, and pq is a prime number.

1. pq is a composite number. We prove that in this case there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the linear forms L_1 and L_2 are independent. Obviously, without loss of generality we can assume that the following statement is true: If $y = \frac{m}{n} \in Y, m, n \in \mathbb{Z}$, $n \neq 0$, then the factorization of n into prime factors contains only such s_j from the factorization of $|s|$ for which $f_{s_j} \in \mathrm{Aut}(X)$. For this reason $sy \in H$ implies that $y \in H$. There exist two possibilities.

A. $|p| > 1, |q| > 1$. Since p and s are relatively prime, and so are q and s, we have $H^{(p)} \neq H$ and $H^{(q)} \neq H$. Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_1) \subset A(G, H^{(p)})$, $\sigma(\lambda_2) \subset A(G, H^{(q)})$. It follows from this that $\hat{\lambda}_1(y) = 1$, $y \in H^{(p)}$, and $\hat{\lambda}_2(y) = 1$, $y \in H^{(q)}$. Hence by Proposition 2.13,

$$\hat{\lambda}_1(u + pv) = \hat{\lambda}_1(u), \quad \hat{\lambda}_2(u + qv) = \hat{\lambda}_2(u), \quad u, v \in H. \tag{15.20}$$

Consider the functions $g_j(y)$ on the group Y of the form

$$g_j(y) = \begin{cases} \hat{\lambda}_j(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases} \tag{15.21}$$

By Proposition 2.12, $g_j(y)$ are positive definite functions on Y. By Bochner's theorem there exist distributions $\mu_j \in \mathrm{M}^1(X)$ such that $\hat{\mu}_j(y) = g_j(y)$, $j = 1, 2$. We will show that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15.19). We conclude from (15.20) and (15.21) that if $u, v \in H$, then equation (15.19) is satisfied. Let $u \notin H$. Then the right-hand side of (15.19) is equal to zero. If the left-hand side of (15.19) is not equal to zero, we have

$$\begin{cases} u + pv \in H, \\ u + qv \in H. \end{cases} \tag{15.22}$$

It follows from (15.22) that $sv \in H$. Therefore $v \in H$, contrary to the choice of v. Thus the left-hand side of (15.19) is also equal to zero. Let $v \notin H$. If $pv, qv \in H$, then $sv \in H$. We deduce from this that $v \in H$, contrary to the choice of v. Hence either $pv \notin H$ or $qv \notin H$. Then the right-hand side of (15.19) is equal to zero. If the left-hand side of (15.19) is not equal to zero, then (15.22) holds. This implies $v \in H$, contrary to the choice of v. Hence the left-hand side of (15.19) is also equal to zero.

We checked that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15.19). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then by Lemma 10.1 the linear forms L_1 and L_2 are independent. It is clear that λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The desired statement in case A is proved.

B. Either $|p| = 1$ or $|q| = 1$. Assume for definiteness that $|p| = 1$. Without loss of generality, we suppose $p = 1$. Let $q = q_1 q_2$ be a decomposition of q, where $|q_j| > 1$, $j = 1, 2$. It is obvious that if $f_q \in \mathrm{Aut}(X)$, then $f_{q_1}, f_{q_2} \in \mathrm{Aut}(X)$. Note that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + q\xi_2$ are independent if and only if the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \frac{1}{q_1}\xi_1 + q_2\xi_2$ are independent. Making the substitution $\zeta_1 = \frac{1}{q_1}\xi_1$, we reduce the problem to the case when $L_1 = q_1\xi_1 + \xi_2$, $L_2 = \xi_1 + q_2\xi_2$.

Equation 10.1 (i) for the characteristic functions $\hat{\mu}_j(y)$ in this case takes the form

$$\hat{\mu}_1(q_1 u + v)\hat{\mu}_2(u + q_2 v) = \hat{\mu}_1(q_1 u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(q_2 v), \quad u, v \in Y. \tag{15.23}$$

Assume that $\lambda_j \in M^1(G)$ and $\sigma(\lambda_j) \subset A(G, H^{(q_j)})$, $j = 1, 2$. It is obvious that $\hat{\lambda}_j(y) = 1$, $y \in H^{(q_j)}$. Hence by Proposition 2.13,

$$\hat{\lambda}_1(q_1 u + v) = \hat{\lambda}_1(v), \quad \hat{\lambda}_2(u + q_2 v) = \hat{\lambda}_2(u) \quad u, v \in H. \tag{15.24}$$

In the same manner as in case A we define the functions $g_j(y)$ by formulas (15.21) and the distributions $\mu_j \in \mathrm{M}^1(X)$. To show that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15.23) we reason as in case A. If $u, v \in H$, in view of (15.24) equation (15.23) holds true. Let $u \notin H$. Then the right-hand side of equation (15.23) is equal to zero. If the left-hand side of (15.23) is not equal to zero, then we have

$$\begin{cases} q_1 u + v \in H, \\ u + q_2 v \in H. \end{cases}$$

It follows from this that $su \in H$, and hence $u \in H$, contrary to the choice of u. Thus the left-hand side of (15.23) is also equal to zero. The case $v \notin H$ is considered similarly. We checked that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15.23). If ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then by Lemma 10.1 the linear forms L_1 and L_2 are independent. Since the distributions λ_j can be chosen in such a way that $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$, the desired assertion in case B is proved. Therefore, the statement of the theorem in case 1 is proved.

2. $pq = -1$. Assume for definiteness that $p = 1$. Then $q = -1$ and L_1 and L_2 have the form $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 - \xi_2$. It follows from Theorem 7.10, if $f_2 \in \mathrm{Aut}(X)$, then the independence of L_1 and L_2 implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$. If $f_2 \notin \mathrm{Aut}(X)$, then there exist independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that the linear forms L_1 and L_2 are independent.

3. pq is a prime number. Assume for definiteness that $p = 1$ and q is a prime number, i.e., $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + q\xi_2$. Equation 10.1 (i) for the characteristic functions $\hat{\mu}_j(y)$ takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + qv) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(qv), \quad u, v \in Y. \tag{15.25}$$

There exist the following possibilities.

A. Either $f_{s_j} \notin \mathrm{Aut}(X)$ at least for one $s_j \neq 2$ or $f_2 \notin \mathrm{Aut}(X)$ and s is divisible by 4. We will prove that there exist independent random variables ξ_1 and ξ_2 with values in the group X and distributions $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$ such that L_1 and L_2 are independent. Assume that $f_{s_j} \notin \mathrm{Aut}(X)$ for some $s_j \neq 2$. This implies that $Y^{(s_j)} \neq Y$. Take $y_0 \notin Y^{(s_j)}$ and consider on the group X the function

$$\rho(x) = 1 + \mathrm{Re}(x, y_0).$$

Then $\rho(x) \geq 0$, $x \in X$, and

$$\int\limits_X \rho(x)\, dm_X(x) = 1.$$

Let $\mu \in M^1(X)$ be the distribution with density $\rho(x)$ with respect to the Haar distribution m_X. We have

$$\hat{\mu}(y) = \begin{cases} 1 & \text{if } y = 0, \\ \frac{1}{2} & \text{if } y = \pm y_0, \\ 0 & \text{if } y \notin \{0, \pm y_0\}. \end{cases}$$

Let ξ_1 and ξ_2 be independent identically distributed random variables with values in X and distribution μ. We will verify that the characteristic functions of the random variables ξ_j satisfy equation (15.25) which becomes the form

$$\hat{\mu}(u+v)\hat{\mu}(u+qv) = \hat{\mu}^2(u)\hat{\mu}(v)\hat{\mu}(qv), \quad u, v \in Y. \tag{15.26}$$

Obviously, it suffices to verify that equation (15.26) holds true when $u \neq 0$, $v \neq 0$. In this case the right-hand side of (15.26) is equal to zero. If the left-hand side of (15.26) is not equal to zero, then

$$\begin{cases} u + v \in \{0, \pm y_0\}, \\ u + qv \in \{0, \pm y_0\}. \end{cases}$$

This implies that

$$sv \in \{0, \pm y_0, \pm 2y_0\}. \tag{15.27}$$

Since the numbers 2 and s_j are relatively prime and $y_0 \notin Y^{(s_j)}$, we have $2y_0 \notin Y^{(s_j)}$, but this contradicts (15.27). The contradiction obtained shows that the left-hand side of (15.25) is also equal to zero.

In the case where $f_2 \notin \mathrm{Aut}(X)$ and s is divisible by 4 we reason similarly. We take $y_0 \notin Y^{(2)}$ and obtain the contradiction with (15.27) because s is divisible by 4.

Hence, the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (15.26). Therefore, by Lemma 10.1, the linear forms L_1 and L_2 are independent. By the construction $\mu_1, \mu_2 \notin \Gamma(X) * I(X)$. The theorem is proved in case A.

B. $f_s \in \mathrm{Aut}(X)$. We will prove that in this case at least one of the distributions $\mu_j \in \Gamma(X) * I(X)$. Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that

$\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (15.25). Set $a(y) = \hat{\nu}_1(y)$, $b(y) = \hat{\nu}_2(y)$. It is clear that the functions $a(y)$ and $b(y)$ satisfy equation 15.1 (i) and conditions 15.1 (ii). We deduce from equation 15.1 (i) that the functions $a(y)$ and $b(y)$ satisfy equations (15.1) and (15.2).

Assume first that $a(y)b(y) = 0$, $y \in Y$, $y \neq 0$. It follows from (15.1) that $a(sy) = 0$, $y \in Y$, $y \neq 0$. Since $f_s \in \mathrm{Aut}(X)$, we conclude that $a(y) = 0$, $y \in Y$, $y \neq 0$. Hence $\nu_1 = m_X$, so that $\mu_1 = m_X$. Similarly, (15.2) implies that $\mu_2 = m_X$.

Assume now that $a(y_0)b(y_0) \neq 0$ for some $y_0 \in Y$, $y_0 \neq 0$. Consider two cases: $q > 0$ and $q < 0$.

(i) $q > 0$. Since $Y^{(s)} = Y$, we can apply Lemma 15.1 and obtain a subgroup $M \subset Y$ such that $a(y)b(y) \neq 0$, $y \in M$. The restrictions of the characteristic functions $a(y)$ and $b(y)$ to M satisfy the conditions of Corollary 15.3. Therefore we have representation 15.3 (i) on $M^{(sq)}$. Substituting representation 15.3 (i) into 15.1 (i) we get $\sigma_1 + q\sigma_2 = 0$. This implies that $\sigma_1 = \sigma_2 = 0$. Thus on the subgroup $M^{(sq)}$ the equality $a(y) = b(y) = 1$ is fulfilled. Put $E = \{y \in Y : a(y) = b(y) = 1\}$. Then E is a nonzero subgroup of Y and by Proposition 2.13 the following equalities hold:

$$a(y+h) = a(y), \quad b(y+h) = b(y), \quad y \in Y,\ h \in E.$$

Therefore we can pass from equation 15.1 (i) on the group Y to the induced equation on the factor group $L = Y/E$ putting $\hat{a}([y]) = a(y)$, $\hat{b}([y]) = b(y)$, $y \in [y]$. We note that

$$\{[y] \in L : \hat{a}([y]) = \hat{b}([y]) = 1\} = \{0\}. \tag{15.28}$$

We deduce from (15.2) that f_s is a monomorphism on L. Since L is a torsion group, $f_s \in \mathrm{Aut}(L)$. Taking into account that $\hat{a}([y]) \leq 1$, $\hat{b}([y]) \leq 1$, it follows from (15.1) that

$$\hat{a}(s[y]) \leq \hat{a}([y]), \quad [y] \in L. \tag{15.29}$$

Since L is a torsion group, we have $s^n[y] = [y]$ for every $[y] \in L$ and some natural n. Generally, n depends on $[y]$. Inequality (15.29) implies that

$$\hat{a}([y]) = \hat{a}(s^n[y]) \leq \cdots \leq \hat{a}(s[y]) \leq \hat{a}([y]), \quad [y] \in L.$$

Hence the function $\hat{a}([y])$ takes a constant value on each orbit $\{[y], s[y], \ldots, s^{n-1}[y]\}$. The analogous statement for the function $\hat{b}([y])$ follows from (15.2).

Assume that $\hat{a}([y]_0) \neq 0$ for some $[y]_0 \in L$, $[y]_0 \neq 0$. It follows from this that $\hat{a}(s[y]_0) = \hat{a}([y]_0) \neq 0$ and (15.1) implies that

$$\hat{a}(q[y]_0) = \hat{b}(q[y]_0) = 1. \tag{15.30}$$

We conclude from (15.28) and (15.30) that $q[y]_0 = 0$. Arguing as above we find from (15.2) that if $\hat{b}([y]_0) \neq 0$, $[y]_0 \in L$, $[y]_0 \neq 0$, then $\hat{a}([y]_0) = \hat{b}(q[y]_0) = 1$. Hence (15.30) is also fulfilled. Note that any factor group of the group Y contains no more then one subgroup $A \cong \mathbb{Z}(q)$. In particular, it is true for the factor group L. Let A be the subgroup of L generated by the element $[y]_0$. Consider the restriction of equation 15.1 (i) to A. Taking into account that $q[y] = 0$ for all $[y] \in A$, we get

$$\hat{a}([u] + [v])\hat{b}([u]) = \hat{a}([u])\hat{b}([u])\hat{a}([v]), \quad [u], [v] \in A. \tag{15.31}$$

Assume that $\hat{b}([u]_0) \neq 0$ for $[u]_0 \in A$, $[u]_0 \neq 0$. It follows from (15.31) that

$$\hat{a}([u]_0 + [v]) = \hat{a}([u]_0)\hat{a}([v]), \quad [v] \in A.$$

Substituting here $[v] = (q-1)[u]_0$, we obtain $\hat{a}([u]_0) = 1$. Since q is a prime number, $\hat{a}([y]) = 1$ for $[y] \in A$. In the case when $\hat{b}([y]) = 0$ for all $[y] \in A$, $[y] \neq 0$ it is obvious that the $\hat{a}([y])$ can be an arbitrary function. Thus we have proved that either

$$\hat{a}([y]) = \begin{cases} 1 & \text{if } [y] \in A, \\ 0 & \text{if } [y] \notin A \end{cases} \tag{15.32}$$

or

$$\hat{b}([y]) = \begin{cases} 1 & \text{if } [y] = 0, \\ 0 & \text{if } [y] \neq 0. \end{cases} \tag{15.33}$$

Returning from the induced functions $\hat{a}([y])$ and $\hat{b}([y])$ on L to the original functions $a(y)$ and $b(y)$ on Y, we get that either $\nu_1 \in I(X)$ or $\nu_2 \in I(X)$. Hence either $\mu_1 \in I(X)$ or $\mu_2 \in I(X)$.

In the case when $\hat{a}([y]_0) = \hat{b}([y]_0) = 0$ for all $[y]_0 \in L$, $[y]_0 \neq 0$, representation (15.32) holds for the function $\hat{a}([y])$, where $A = \{0\}$, and representation (15.33) holds for the function $\hat{b}([y])$. Obviously, in this case $\mu_1, \mu_2 \in I(X)$. Case (i), $q > 0$, is completely studied.

(ii) $q < 0$. We conclude from (15.2) that

$$a\left(\frac{y_0}{s^m}\right) b\left(\frac{y_0}{s^m}\right) \neq 0$$

for any natural m. It follows from the proof of Lemma 15.1 that $a(y)b(y) \neq 0$ on the subgroup $y_0 H^{(q)}$. Hence by Corollary 15.3 representation 15.3 (i) holds on the subgroup $y_0 H^{(q^2)}$. Let F be the maximal subgroup of Y containing $y_0 H^{(q^2)}$ on which representation 15.3 (i) holds.

Since $F \not\cong \mathbb{Z}$, the subgroup F is dense in $\mathbb{R}$ in the topology of $\mathbb{R}$. Using 15.3 (i) we can apply Lemma 15.5 to the functions $a(y)$ and $b(y)$. We conclude that for any $u \in Y$ there exist the limits

$$a_u = \lim_{y \to 0, y \in u+F} a(y), \quad b_u = \lim_{y \to 0, y \in u+F} b(y).$$

Note that the functions a_u and b_u by the construction take constant values on each coset $u+F$. Therefore they define some functions $a_{[u]}$ and $b_{[u]}$ on the factor group $L = Y/F$. Obviously, the functions $a_{[u]}$ and $b_{[u]}$ satisfy equation 15.1 (i) and conditions 15.1 (ii). Hence they also satisfy equations (15.1) and (15.2). Since F is the maximal subgroup of

Y on which representation 15.3 (i) holds, it follows from statement (c) of Lemma 15.5 that

$$\{[u] \in L : a_{[u]} = b_{[u]} = 1\} = \{0\}.$$

The obtained condition is an analogue of (15.28). Taking into account that L is a torsion group, we deduce from (15.2) that $f_s \in \mathrm{Aut}(L)$. Arguing as in case (i), we obtain either representation (15.32) for the function $a_{[u]}$ or representation (15.33) for the function $b_{[u]}$.

We will prove that if $a_{[u]} = 0$ for all $[u] \notin A$, then $a(y) = 0$ for all $y \notin f_q^{-1}(F)$. Indeed, substitute $u \in F$, $v_0 \notin f_q^{-1}(F)$ in 15.1 (i). Taking into account 15.3 (i), we get

$$a(u + v_0)b(u + qv_0) = e^{-\sigma_1 u^2 - \sigma_2 q^2 u^2} a(v_0)b(qv_0).$$

Passing here to the limit as $u \to -v_0$, $u \in F$ and taking into account that $[v_0] \notin A$, we arrive at

$$0 = e^{-\sigma_1 v_0^2 - \sigma_2 q^2 v_0^2} a(v_0)b(qv_0).$$

Hence $a(v_0)b(qv_0) = 0$. It follows from (15.1) that $a(sv_0) = 0$. Note that $f_s \in \mathrm{Aut}(F)$. Indeed, by Corollary 4.7 we conclude from (15.1) that representation 15.3 (i) holds on the subgroup $f_s^{-1}(F)$. Hence $f_s^{-1}(F) = F$ because F is the maximal subgroup for which representation 15.3 (i) is true. This implies that $f_s(F) = F$, so that $f_s \in \mathrm{Aut}(F)$. We deduce from this that f_s is a one-to-one mapping of the set $f_q^{-1}(F)$ onto itself. Hence if $a(sv) = 0$ for all $v \notin f_q^{-1}(F)$, then $a(v) = 0$ for all $v \notin f_q^{-1}(F)$. Thus we obtain from representation (15.32) for the function $a_{[u]}$ the following representation for the function $a(y)$:

$$a(y) = \begin{cases} e^{-\sigma_1 y^2} & \text{if } y \in f_q^{-1}(F), \\ 0 & \text{if } y \notin f_q^{-1}(F). \end{cases} \tag{15.34}$$

Arguing as above we deduce that if $b_{[u]} = 0$ for all $[u] \neq 0$, then $b(y) = 0$ for all $y \notin F$. In this case representation (15.33) for the function $b_{[u]}$ implies that

$$b(y) = \begin{cases} e^{-\sigma_2 y^2} & \text{if } y \in F, \\ 0 & \text{if } y \notin F. \end{cases} \tag{15.35}$$

Thus at least one of the distributions $\nu_j \in \Gamma(X) * I(X)$. Then by Lemma 15.4 the corresponding distribution $\mu_j \in \Gamma(X) * I(X)$. Case (ii) $q < 0$ is also studied. Case B is completely studied.

C. $f_{s_j} \in \mathrm{Aut}(X)$ for all $s_j \neq 2$, $f_2 \notin \mathrm{Aut}(X)$ and s is not divisible by 4. We will prove that in this case at least one of the distributions $\mu_j \in \Gamma(X) * I(X)$. Since s is divisible by 2 and s is not divisible by 4, we have $q = 4k - 1$, $k \in \mathbb{Z}$. As in case B put $\nu_j = \mu_j * \bar{\mu}_j$, $j = 1, 2$, $a(y) = \hat{\nu}_1(y)$, $b(y) = \hat{\nu}_2(y)$. There are two possibilities.

(i) Assume that

$$a(y)b(y) = 0, \quad y \in Y^{(2)}, \quad y \neq 0. \tag{15.36}$$

Substituting $u = qy$, $v = y$, and then $u = v = y$ into 15.1 (i) and taking into account (15.1) and (15.2), we get

$$a(sy) = a((q+1)y)b(2qy), \quad y \in Y, \tag{15.37}$$
$$b(sy) = a(2y)b((q+1)y)), \quad y \in Y. \tag{15.38}$$

Since $s = 2 - 4k$, it follows from (15.37) and (15.38) that

$$a((1-2k)y) = a(2ky)b(qy), \quad y \in Y, \tag{15.39}$$
$$b((1-2k)y) = a(y)b(2ky), \quad y \in Y. \tag{15.40}$$

Multiplying (15.39) and (15.40), we find

$$a((1-2k)y)b((1-2k)y) = a(2ky)b(2ky)a(y)b(qy), \quad y \in Y. \tag{15.41}$$

Since $1 - 2k$ is a factor of s, we have $f_{1-2k} \in \mathrm{Aut}(X)$. Then (15.41) and (15.36) yields that $a(y)b(y) = 0$, $y \in Y$, $y \neq 0$. In this case we conclude from (15.1) and (15.2) that $a(y) = b(y) = 0$, $y \in Y^{(2)}$, $y \neq 0$. Then (15.39) and (15.40) imply that $a(y) = b(y) = 0$, $y \in Y$, $y \neq 0$. It means that $\nu_1 = \nu_2 = m_X$, and hence $\mu_1 = \mu_2 = m_X$. Case (i) is completely studied.

(ii) Assume that condition (15.36) does not hold, i.e., $a(y_0)b(y_0) \neq 0$ for some $y_0 \in Y^{(2)}$, $y_0 \neq 0$. If $q > 0$ we argue as in case B (i), because the equality $Y^{(s)} = Y^{(2)}$ implies that the functions $a(y)$ and $b(y)$ satisfy the conditions of Lemma 15.1.

If $q < 0$, we follow the scheme of the proof of case B (ii). We also make some remarks because in contrast to case B (ii), $f_s \notin \mathrm{Aut}(X)$. We retain the same notation as in case B (ii).

In order to prove that there exists a subgroup $W \ncong \mathbb{Z}$ on which representation 15.3 (i) holds we reason as follows. Assume that on the subgroup M generated by some element ζ the inequality $a(y)b(y) \neq 0$ is fulfilled (the existence of such a subgroup follows from Lemma 15.1). Substitute

$$u = \frac{l}{1-2k}\zeta, \quad v = \Big(1 - \frac{l}{1-2k}\Big)\zeta, \quad l \in \mathbb{Z},$$

in 15.1 (i). For such u and v we have $u + v, u + qv \in M$. Hence the left-hand side of 15.1 (i) is not equal to zero, so that

$$a\Big(\frac{l}{1-2k}\zeta\Big)\, b\Big(\frac{l}{1-2k}\zeta\Big) \neq 0.$$

Since $f_{1-2k} \in \mathrm{Aut}(X)$, we obtain that the inequality $a(y)b(y) \neq 0$ holds on the subgroup $S = \zeta H$. It follows from $q \neq -1$ that $H \ncong \mathbb{Z}$, so that $S \ncong \mathbb{Z}$. Then we argue as in the proof of case B (ii). We obtain either representation (15.32) for the function $a_{[u]}$ or representation (15.33) for the function $b_{[u]}$. To get the desired representations, i.e., either (15.34) for the function $a(y)$ or (15.35) for the function $b(y)$, it suffices to show that if $a(sy) = 0$ for all $y \notin f_q^{-1}(F)$, then $a(y) = 0$, $y \notin f_q^{-1}(F)$, and also if $b(sy) = 0$ for all $y \notin F$, then $b(y) = 0$, $y \notin F$.

We will give the proof only for the function $a(y)$. For the function $b(y)$ we argue similarly. Take $y_0 \notin f_q^{-1}(F)$. Consider the coset $y_0 + F$ and notice that $q[y_0] \neq 0$. It follows from $f_s \in \mathrm{Aut}(L)$ that

$$s[y'] = [y_0] \tag{15.42}$$

for some $[y'] \in L$. We note that $y' \notin f_q^{-1}(F)$ because in the opposite case $q[y'] = 0$, and taking into account (15.42), we have $[y'] = [y_0]$, contrary to the choice of y_0. We conclude from (15.42) that $sy' = y_0 + h, h \in F$. This implies that $s(y' + h') \in y_0 + F$ for all $h' \in F$. Since $F \ncong \mathbb{Z}$, the set $s(y' + F)$ is dense in $y_0 + F$ in the topology of $\mathbb{R}$. The function $a(y)$ is uniformly continuous on $y_0 + F$ in the topology of $\mathbb{R}$ and by the condition $a(s(y' + h')) = 0$ for any $h' \in F$. Hence $a(y) \equiv 0$, $y \in y_0 + F$, in particular $a(y_0) = 0$. Case (ii) is completely studied. The desired statement in case C is proved, hence the statement of the theorem in case 3 is also proved. □

Remark 15.8. It follows from the proof of Theorem 15.7 in cases 3B and 3C that equation (15.25) always has solutions $\hat{\mu}_j(y)$ such that one of the distributions $\mu_j \notin \Gamma(X) * I(X)$. Hence the statements of Theorem 15.7 in cases 3B and 3C may not be strengthened.

Remark 15.9. Let X be a connected group, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. We conclude from Theorems 1.9.6 and 7.10 that the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$ if and only if the condition

$$\text{(i)} \quad f_2 \in \mathrm{Aut}(X)$$

is satisfied. It follows from Theorem 15.7 that the following statements hold true:

(α) Let $X = \Sigma_{\boldsymbol{a}}$. Then (i) is a necessary condition in order that, for some automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, the independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ imply that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$.

(β) Let $X = \Sigma_{\boldsymbol{a}}$ and $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. Assume that condition (i) is fulfilled. The only forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ for which the independence of L_1 and L_2 implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$ are $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$.

Generally, statements (α) and (β) are not true for arbitrary compact connected Abelian groups. To construct the corresponding counterexample for statement (α) put $X = X_1 \times X_2$, where $X_j = \Sigma_{\boldsymbol{a}_j}$, $\boldsymbol{a}_1 = (2, 2, \dots)$, $\boldsymbol{a}_2 = (3, 3, \dots)$. Obviously, the group X is connected and $f_2 \notin \mathrm{Aut}(X)$. Let an automorphism $\alpha \in \mathrm{Aut}(X)$ be of the form $\alpha(x_1, x_2) = (-x_1, x_2)$, $x_j \in X_j$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. We verify that if the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$ are independent, then $\mu_1, \mu_2 \in \Gamma(X) * I(X)$.

By Lemma 10.1 the linear forms L_1 and L_2 are independent if and only if the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + \tilde{\alpha}v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(\tilde{\alpha}v), \quad u, v \in Y, \tag{15.43}$$

where $\tilde{\alpha}(y_1, y_2) = (-y_1, y_2)$, $y_j \in Y_j = X_j^*$. Set $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. The characteristic functions

$\hat{\nu}_j(y)$ also satisfy equation (15.43). Consider the restriction of equation (15.43) for the functions $\hat{\nu}_j(y)$ to the subgroup Y_2. We have

$$\hat{\nu}_1(u+v)\hat{\nu}_2(u+v) = \hat{\nu}_1(u)\hat{\nu}_2(u)\hat{\nu}_1(v)\hat{\nu}_2(v), \quad u, v \in Y_2. \tag{15.44}$$

Put $g(y) = \hat{\nu}_1(y)\hat{\nu}_2(y)$. We conclude from (15.44) that $g(y) = 1$, $y \in Y_2$. Hence $\hat{\nu}_j(y) = 1$, $y \in Y_2$, $j = 1, 2$. By Proposition 2.13, $\sigma(\nu_j) \subset A(X, Y_2) = X_1$. Applying Proposition 2.2 we obtain that the distributions μ_j can be substituted by their shifts μ'_j in such a way that $\sigma(\mu'_j) \subset X_1$. The characteristic functions of the distributions μ'_j also satisfy equation (15.43). Since $\hat{\mu}'_j(y+h) = \hat{\mu}'_j(y)$ for all $y \in Y$, $h \in Y_2$, it suffices to solve equation (15.43) assuming that $u, v \in Y_1$. But on the subgroup Y_1 equation (15.43) takes the form

$$\hat{\mu}'_1(u+v)\hat{\mu}'_2(u-v) = \hat{\mu}'_1(u)\hat{\mu}'_2(u)\hat{\mu}'_1(v)\hat{\mu}'_2(-v), \quad u, v \in Y_1. \tag{15.45}$$

Note that $f_2 \in \mathrm{Aut}(X_1)$ and apply Theorem 7.10 to the group X_1. We deduce from (15.45) that $\mu'_j \in \Gamma(X_1) * I(X_1)$. This implies that $\mu_j \in \Gamma(X) * I(X)$.

We will obtain a counterexample to statement (β) if in the considerations given above we take $X = X_1^2$.

Chapter VI

The Heyde theorem for locally compact Abelian groups

Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, α_j, β_j be nonzero real numbers. Assume that the conditions $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$ are satisfied. By the Heyde theorem if the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all random variables ξ_j are Gaussian. Let X be a second countable locally compact Abelian group, ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables with values in X and distributions μ_j. This chapter is devoted to some group analogues of the Heyde theorem. We consider linear forms of independent random variables taking values in X, and coefficients of the linear forms are topological automorphisms of X. First we assume that the characteristic functions of the considered random variables do not vanish. Then we omit this restriction but suppose that X is either a finite or a discrete group.

16 The characteristic functions of random variables do not vanish

Let X be a second countable locally compact Abelian group, Y be its character group. In this section we describe groups X that have the following property: if α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, are topological automorphisms of the group X satisfying the conditions $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$, and ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, are independent random variables with values in X and distributions μ_j with non-vanishing characteristic functions, then the symmetry of the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ implies that all $\mu_j \in \Gamma(X)$. We prove that it holds if and only if X contains no elements of order 2. Thus if a group X contains elements of order 2, a natural problem arises: to describe all possible distributions μ_j of independent random variables ξ_j assuming that the conditional distribution of the linear form L_2 given L_1 is symmetric. We solve this problem for two independent random variables taking values in the two-dimensional torus $\mathbb{T}^2$.

Lemma 16.1. *Let α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, be topological automorphisms of a group X. Let ξ_j be independent random variables with values in X and distributions μ_j. The conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric if and only if the characteristic functions $\hat\mu_j(y)$*

satisfy the equation

$$\text{(i)} \quad \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u - \tilde{\beta}_j v), \quad u, v \in Y.$$

Proof. Taking into account that $\hat{\mu}_j(y) = \mathbf{E}[(\xi_j, y)]$, and the random variables ξ_j are independent, we have

$$\begin{aligned}
\text{(i)} \Leftrightarrow & \prod_{j=1}^{n} \mathbf{E}[(\xi_j, \tilde{\alpha}_j u + \tilde{\beta}_j v)] = \prod_{j=1}^{n} \mathbf{E}[(\xi_j, \tilde{\alpha}_j u - \tilde{\beta}_j v],\\
\Leftrightarrow & \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u + \tilde{\beta}_j v)\Big] = \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u - \tilde{\beta}_j v)\Big],\\
\Leftrightarrow & \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u) \prod_{j=1}^{n} (\xi_j, \tilde{\beta}_j v)\Big] = \mathbf{E}\Big[\prod_{j=1}^{n} (\xi_j, \tilde{\alpha}_j u) \prod_{j=1}^{n} (\xi_j, -\tilde{\beta}_j v)\Big],\\
\Leftrightarrow & \mathbf{E}\Big[\prod_{j=1}^{n} (\alpha_j \xi_j, u) \prod_{j=1}^{n} (\beta_j \xi_j, v)\Big] = \mathbf{E}\Big[\prod_{j=1}^{n} (\alpha_j \xi_j, u) \prod_{j=1}^{n} (-\beta_j \xi_j, v)\Big],\\
\Leftrightarrow & \mathbf{E}[(L_1, u)(L_2, v)] = \mathbf{E}[(L_1, u)(-L_2, v)], \quad u, v \in Y.
\end{aligned} \tag{16.1}$$

Let $(\Omega, \mathfrak{A}, P)$ be a probabilistic space, where the random variables ξ_j are defined. Equality (16.1) is equivalent to the statement that the random variables (L_1, L_2) and $(L_1, -L_2)$ are identically distributed, i.e.,

$$\mathbf{P}\{\omega \in \Omega : L_1(\omega) \in A,\ L_2(\omega) \in B\} = \mathbf{P}\{\omega \in \Omega : L_1(\omega) \in A,\ L_2(\omega) \in -B\}$$

for all $A, B \in \mathfrak{B}(X)$. In view of 2.21 the equality obtained is equivalent to the symmetry of the conditional distribution of L_2 given L_1. □

Equation (i) is called the *Heyde functional equation*.

Theorem 16.2. *Let X be a group containing no elements of order 2, α_j, β_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be topological automorphisms of X such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables taking values in the group X and having distributions μ_j with non-vanishing characteristic functions. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all $\mu_j \in \Gamma(X)$.*

Proof. We can put $\zeta_j = \alpha_j \xi_j$ and reduce the proof of the theorem to the case when $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, $\delta_j \in \mathrm{Aut}(X)$. The conditions $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$ are transformed into the conditions $\delta_i \pm \delta_j \in \mathrm{Aut}(X)$ for all $i \neq j$. By Lemma 16.1 the symmetry of the conditional distribution of $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$ given $L_1 = \xi_1 + \cdots + \xi_n$ implies that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i) which takes the form

$$\prod_{j=1}^{n} \hat{\mu}_j(u + \tilde{\delta}_j v) = \prod_{j=1}^{n} \hat{\mu}_j(u - \tilde{\delta}_j v), \quad u, v \in Y. \tag{16.2}$$

Set $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (16.2). Put $\varphi_j(y) = -\ln \hat{\nu}_j(y)$. We conclude from (16.2) that the functions $\varphi_j(y)$ satisfy the equation

$$\sum_{j=1}^{n}[\varphi_j(u + \tilde{\delta}_j v) - \varphi_j(u - \tilde{\delta}_j v)] = 0, \quad u, v \in Y. \tag{16.3}$$

We use the finite difference method to solve equation (16.3). Let k_1 be an arbitrary element of the group Y. Set $h_1 = \tilde{\delta}_n k_1$, then $h_1 - \tilde{\delta}_n k_1 = 0$. Substitute $u + h_1$ for u and $v + k_1$ for v in equation (16.3). Subtracting equation (16.3) from the resulting equation we obtain

$$\sum_{j=1}^{n} \Delta_{l_{1,j}} \varphi_j(u + \tilde{\delta}_j v) - \sum_{j=1}^{n-1} \Delta_{l_{1,j+n}} \varphi_j(u - \tilde{\delta}_j v) = 0, \quad u, v \in Y, \tag{16.4}$$

where $l_{1,j} = h_1 + \tilde{\delta}_j k_1 = (\tilde{\delta}_n + \tilde{\delta}_j)k_1$, $j = 1, 2, \ldots, n$, $l_{1,j+n} = h_1 - \tilde{\delta}_j k_1 = (\tilde{\delta}_n - \tilde{\delta}_j)k_1$, $j = 1, 2, \ldots, n-1$. Let k_2 be an arbitrary element of the group Y. Put $h_2 = \tilde{\delta}_{n-1}k_2$. Then $h_2 - \tilde{\delta}_{n-1}k_2 = 0$. Substitute $u + h_2$ for u and $v + k_2$ for v in equation (16.4). Subtracting equation (16.4) from the resulting equation we obtain

$$\sum_{j=1}^{n} \Delta_{l_{2,j}} \Delta_{l_{1,j}} \varphi_j(u + \tilde{\delta}_j v) - \sum_{j=1}^{n-2} \Delta_{l_{2,j+n}} \Delta_{l_{1,j+n}} \varphi_j(u - \tilde{\delta}_j v) = 0, \quad u, v \in Y, \tag{16.5}$$

where $l_{2,j} = h_2 + \tilde{\delta}_j k_2 = (\tilde{\delta}_n + \tilde{\delta}_j)k_2$, $j = 1, 2, \ldots, n$, $l_{2,j+n} = h_2 - \tilde{\delta}_j k_2 = (\tilde{\delta}_n - \tilde{\delta}_j)k_2$, $j = 1, 2, \ldots, n-2$. Arguing as above in n steps we get the equation

$$\sum_{j=1}^{n} \Delta_{l_{n,j}} \Delta_{l_{n-1,j}} \ldots \Delta_{l_{1,j}} \varphi_j(u + \tilde{\delta}_j v) = 0, \quad u, v \in Y, \tag{16.6}$$

where $l_{p,j} = (\tilde{\delta}_{n-p+1} + \tilde{\delta}_j)k_p$, $p = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$. Let k_{n+1} be an arbitrary element of the group Y. Set $h_{n+1} = -\tilde{\delta}_n k_{n+1}$, hence $h_{n+1} + \tilde{\delta}_n k_{n+1} = 0$. Substitute $u + h_n$ for u and $v + k_n$ for v in equation (16.6). Subtracting equation (16.6) from the resulting equation we obtain

$$\sum_{j=1}^{n-1} \Delta_{l_{n+1,j}} \Delta_{l_{n,j}} \Delta_{l_{n-1,j}} \ldots \Delta_{l_{1,j}} \varphi_j(u + \tilde{\delta}_j v) = 0, \quad u, v \in Y, \tag{16.7}$$

where $l_{n+1,j} = h_{n+1} + \tilde{\delta}_j k_{n+1} = (\tilde{\delta}_j - \tilde{\delta}_n)k_{n+1}$, $j = 1, 2, \dots, n-1$. Equation (16.7) does not contain the function φ_n. Arguing similarly we sequentially exclude the functions $\varphi_{n-1}, \varphi_{n-2}, \dots, \varphi_2$ from equation (16.7). Finally we obtain

$$\Delta_{l_{2n-1,1}} \Delta_{l_{2n-2,1}} \dots \Delta_{l_{1,1}} \varphi_1(u + \tilde{\delta}_1 v) = 0, \quad u, v \in Y, \tag{16.8}$$

where $l_{p,1} = (\tilde{\delta}_1 + \tilde{\delta}_{n-p+1})k_p$, $p = 1, 2, \dots, n-1$, $l_{n,1} = 2\tilde{\delta}_1 k_n$, $l_{n+p,1} = (\tilde{\delta}_1 - \tilde{\delta}_{n-p+1})k_{n+p}$, $p = 1, 2, \dots, n-1$.

Taking into account that k_p are arbitrary elements of the group Y and $\tilde{\delta}_i \pm \tilde{\delta}_j \in \mathrm{Aut}(Y)$ for $i \neq j$, we can substitute in (16.8) $l_{n,1} = 2k$, $l_{p,1} = h$, $p = 1, 2, \dots, n-1, n+1, \dots, 2n-1$, where k and h are arbitrary elements of the group Y. Substituting $v = 0$ into the resulting equation we obtain

$$\Delta_{2k} \Delta_h^{2n-2} \varphi_1(u) = 0, \tag{16.9}$$

where k, h, and u are arbitrary elements of the group Y. Since the group X contains no elements of order 2, by Theorem 1.9.5 the subgroup $Y^{(2)}$ is dense in Y. We deduce from (16.9) that

$$\Delta_h^{2n-1} \varphi_1(u) = 0, \quad u, h \in Y,$$

i.e., the function $\varphi_1(y)$ is a continuous polynomial. Since the group X contains no elements of order 2, X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. By Theorem 5.11 $\nu_1 \in \Gamma(X)$. Applying Theorem 4.6 we get $\mu_1 \in \Gamma(X)$. Arguing as above we prove that all $\mu_j \in \Gamma(X)$. □

Remark 16.3. The statement of Theorem 16.2 is not valid if a group X contains elements of order 2. To prove this observe that if K is a closed subgroup of a group X, $\mu \in \mathrm{M}^1(X)$, and $\sigma(\mu) \subset K$, then $\hat{\mu}(y + l) = \hat{\mu}(y)$ for $y \in Y, l \in A(Y, K)$. Let $G = X_{(2)}$, i.e., G is a closed subgroup of X generated by all elements of order 2. Taking into account that $A(Y, G) = \overline{Y^{(2)}}$ by Theorem 1.9.5, we see that if $\sigma(\mu) \subset G$, then $\hat{\mu}(y + 2h) = \hat{\mu}(y)$. Hence $\hat{\mu}(y + h) = \hat{\mu}(y - h)$ for all $y, h \in Y$. It follows from this that if ξ_j are independent random variables with values in the subgroup $G \subset X$ and distributions μ_j, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i). By Lemma 16.1 the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ is symmetric. Obviously, if $G \neq \{0\}$, we can suppose that $\mu_j \notin \Gamma(X)$, $j = 1, 2, \dots, n$.

Remark 16.4. Generally, the statement of Theorem 16.2 is not valid, even when a group X contains no elements of order 2, if we assume that the characteristic functions of the considering distributions can vanish. To construct an example let X be the $\boldsymbol{a}$-adic solenoid $X = \Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (2, 3, 4, \dots)$. Then X is a torsion-free group, and $Y \cong \mathbb{Q}$. We will assume that $Y = \mathbb{Q}$.

Take arbitrary nondegenerate distributions $\lambda_1, \lambda_2 \in \mathrm{M}^1(\mathbb{T})$ such that $\sigma(\lambda_1) \subset \mathbb{Z}(2)$, $\sigma(\lambda_2) \subset \mathbb{Z}(3)$. It follows from this that

$$\hat{\lambda}_1(u \pm 2v) = \hat{\lambda}_1(u), \quad \hat{\lambda}_2(u \pm 3v) = \hat{\lambda}_2(u), \quad u, v \in \mathbb{Z}. \tag{16.10}$$

Consider on the group $\mathbb{Q}$ the functions $f_j(y)$ of the form

$$f_j(y) = \begin{cases} \hat{\lambda}_j(y) & \text{if } y \in \mathbb{Z}, \\ 0 & \text{if } y \notin \mathbb{Z}. \end{cases} \tag{16.11}$$

Since $\hat{\lambda}_j(y)$ are positive definite functions on the group $\mathbb{Z}$, by Theorem 2.12, $f_j(y)$ are positive definite functions on the group $\mathbb{Q}$. By the Bochner theorem there exist distributions $\mu_j \in \mathrm{M}^1(\Sigma_{\boldsymbol{a}})$ such that $\hat{\mu}_j(y) = f_j(y)$, $j = 1, 2$. Obviously, $\mu_j \notin \Gamma(\Sigma_{\boldsymbol{a}})$, $j = 1, 2$.

We verify that the characteristic functions $f_j(y)$ satisfy the equation

$$f_1(u + 2v)f_2(u + 3v) = f_1(u - 2v)f_2(u - 3v), \quad u, v \in \mathbb{Q}. \tag{16.12}$$

If $u, v \in \mathbb{Z}$, we deduce from (16.10) and (16.11) that equation (16.12) is satisfied. If $u \notin \mathbb{Z}$, $v \in \mathbb{Z}$, then $u \pm 2v \notin \mathbb{Z}$ and both sides of equation (16.12) are equal to zero. If $v \notin \mathbb{Z}$, then both sides of equation (16.12) are equal to zero. Indeed, if the left-hand side of equation (16.12) is not equal to zero, then $u + 2v, u + 3v \in \mathbb{Z}$. We conclude from this that $v \in \mathbb{Z}$, contrary to the assumption. Arguing similarly we prove that the right-hand side of equation (16.12) is also equal to zero. Thus the characteristic functions $f_j(y)$ satisfy equation (16.12).

Let ξ_1 and ξ_2 be independent random variables with values in the group $X = \Sigma_{\boldsymbol{a}}$ and distributions μ_1 and μ_2. It follows from 1.14 (d) that multiplication by any nonzero integer is a topological automorphism of the group $\Sigma_{\boldsymbol{a}}$. By Lemma 16.1 the conditional distribution of the linear form $L_2 = 2\xi_1 + 3\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.

Assume that a group X contains elements of order 2. As has been noted in Remark 16.3, Theorem 16.2 is false. Therefore for such groups the following natural problem arises: to describe all possible distributions of independent random variables ξ_j with non-vanishing characteristic functions, assuming that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric. Now we solve this problem for two independent random variables taking values in the two-dimensional torus $\mathbb{T}^2$. We need two lemmas.

Lemma 16.5. *Let* $\varepsilon = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Aut}(\mathbb{Z}^2)$. *Assume that* $|\det \varepsilon| = |\det(I \pm \varepsilon)| = 1$. *Consider the equation*

$$\text{(i)} \quad A + B\varepsilon = 0,$$

where $A = (a_{ij})_{i,j=1}^2$ *and* $B = (b_{ij})_{i,j=1}^2$ *are symmetric positive semidefinite matrices. Then*

$$A = \sigma \begin{pmatrix} t_0^2 & t_0 \\ t_0 & 1 \end{pmatrix}, \quad B = kA,$$

where $\sigma \geq 0$, $t_0 = \frac{a-d-\sqrt{5}}{2b}$, $k = \frac{\sqrt{5}+a+d}{2}$.

Proof. It follows from $|\det \varepsilon| = |\det(I \pm \varepsilon)| = 1$ that $\det \varepsilon = -1$ and $|a + d| = 1$. The conditions of item 2A of the proof of Lemma 11.2 are satisfied. The desired representation follows from (11.27). □

Lemma 16.6. *Let a group X be of the form $X = \mathbb{R}^m \times G$, where $m \geq 1$ and all nonzero elements of the group G have order 2. Assume that $\mu \in \mathrm{M}^1(X)$, $\mu = \lambda * \omega$, where $\lambda \in \Gamma(\mathbb{R}^m)$, $\omega \in \mathrm{M}^1(G)$, and the characteristic function of ω does not vanish. Then any factor μ_1 of the distribution μ can be represented in the form $\mu_1 = \tau * \rho$, where $\tau \in \Gamma(\mathbb{R}^m)$ and $\rho \in \mathrm{M}^1(G)$.*

Proof. We will prove the lemma only for the case $m = 1$, i.e., $X = \mathbb{R} \times G$. The case when $m > 1$ can be considered similarly. We have $Y \cong \mathbb{R} \times H$, where $H = G^*$. To avoid introducing new notation we will assume that $Y = \mathbb{R} \times H$. Denote by (s, h), $s \in \mathbb{R}$, $h \in H$, elements of the group Y. We can assume without loss of generality that $\hat{\lambda}(s) = \exp\{-\sigma s^2\}$. We note that the characteristic function of any distribution on the group G is real-valued. Therefore the characteristic function $\hat{\omega}(h)$ can be represented in the form $\hat{\omega}(h) = \exp\{-d(h) + i\pi k(h)\}$, $h \in H$, where $d(h) \geq 0$, $k(h) \in \mathbb{Z}$. In view of 2.7 (c) we have

$$\hat{\mu}(s, h) = \exp\{-\sigma s^2 - d(h) + i\pi k(h)\}, \quad (s, h) \in Y.$$

Let μ_1 be a factor of μ, i.e., $\mu = \mu_1 * \mu_2$, where $\mu_j \in \mathrm{M}^1(X)$, $j = 1, 2$. It follows from 2.7 (c) that

$$\hat{\mu}(s, h) = \hat{\mu}_1(s, h)\hat{\mu}_2(s, h), \quad (s, h) \in Y. \tag{16.13}$$

Note that $\hat{\mu}(s, 0)$ is an entire function. This implies that $\hat{\mu}_1(s, 0)$ is also an entire function (see [74], Theorem 6.2.1). By Proposition 2.20 the function $\hat{\mu}_1(s, h)$ is an entire function for any fixed $h \in H$, and equality (16.13) holds for all $s \in \mathbb{C}$, $h \in H$. Therefore we can represent the function $\hat{\mu}_1(s, h)$ in the form

$$\hat{\mu}_1(s, h) = \exp\{f(s, h)\}, \quad s \in \mathbb{C},\ h \in H,$$

where a branch of the entire function $\ln \hat{\mu}_1(s, h)$ is chosen in such a way that $f(s, h) = \overline{f(-s, h)}$. Substituting $h = 0$ into (16.13) and applying Theorem 2.18 we obtain

$$\hat{\mu}_1(s, 0) = \exp\{-as^2 + ibs\}, \tag{16.14}$$

where $0 \leq a \leq \sigma$, $b \in \mathbb{R}$. Note that

$$\max_{|s| \leq r, h \in H} |\hat{\mu}_1(s, h)| = \max_{|s| \leq r} |\hat{\mu}_1(s, 0)| = \exp\{ar^2 + |b|r\}.$$

Hence $\operatorname{Re} f(s, h) = O(r^2)$, and this means that $f(s, h)$ is a polynomial of degree ≤ 2. We have

$$\hat{\mu}_1(s, h) = \exp\{a(h)s^2 + b(h)s + c(h)\}. \tag{16.15}$$

Let $\varrho \in \mathrm{M}^1(X)$ be an arbitrary distribution such that for any fixed $h \in H$ the function $\hat{\varrho}(s, h)$ is an entire function in s. Then the function

$$g_s(h; \varrho) = \frac{\hat{\varrho}(is, h)}{\hat{\varrho}(is, 0)} \tag{16.16}$$

for any fixed $s \in \mathbb{R}$ is the characteristic function of a distribution on G. We conclude from (16.13) and (16.16) that

$$g_s(h;\mu) = g_s(h;\mu_1)g_s(h;\mu_2).$$

Obviously, $g_s(h;\mu) = \widehat{\omega}(h)$. Thus $g_s(h;\mu_1) = \widehat{\omega}_s(h)$, where ω_s is a factor of ω. The characteristic function $\widehat{\omega}_s(h)$ can be written in the form

$$\widehat{\omega}_s(h) = \exp\{-d_s(h) + i\pi k_s(h)\}, \tag{16.17}$$

where $d_s(h) \geq 0, k_s(h) \in \mathbb{Z}$. It follows from (16.16) that the functions $d_s(h)$ and $k_s(h)$ are continuous in s. Since $k_s(h)$ is an integer-valued function, we have $k_s(h) = k_0(h)$. Therefore we deduce from this (16.17) that

$$\widehat{\omega}_s(h) = \exp\{-d_s(h) + i\pi k_0(h)\}. \tag{16.18}$$

Since ω_s is a factor of ω, we have

$$0 \leq d_s(h) \leq d(h). \tag{16.19}$$

We conclude from (16.16) that

$$\hat{\mu}_1(is,h) = g_s(h;\mu_1)\hat{\mu}_1(is,0). \tag{16.20}$$

Taking into account (16.14) and (16.18), it follows from (16.20) that

$$\hat{\mu}_1(is,h) = \exp\{as^2 - bs - d_s(h) + i\pi k_0(h)\}. \tag{16.21}$$

We find from (16.15) and (16.21) that

$$-a(h)s^2 + ib(h)s + c(h) = as^2 - bs - d_s(h) + i\pi k_0(h) + 2\pi i n(s,h), \tag{16.22}$$

where $n(s,h) \in \mathbb{Z}$. Since $n(s,h)$ is an integer-valued function and continuously depends on s we get $n(s,h) = n(h)$. Note that in view of (16.19) the function $d_s(h)$ as a function in s is bounded. Then (16.22) yields that

$$a(h) = -a, \quad b(h) = ib. \tag{16.23}$$

We conclude from (16.15) that $\hat{\mu}_1(0,h) = \exp\{c(h)\}$. Finally we get from (16.23), (16.14), and (16.15) that

$$\hat{\mu}_1(s,h) = \hat{\mu}_1(s,0)\hat{\mu}_1(0,h).$$

Consider a Gaussian distribution τ on the group $\mathbb{R}$ with the characteristic function $\hat{\tau}(s,h) = \hat{\mu}_1(s,0)$ and a distribution ρ on the group G with the characteristic function $\hat{\rho}(s,h) = \hat{\mu}_1(0,h)$. Since $\hat{\mu}_1(s,h) = \hat{\tau}(s,h)\hat{\rho}(s,h)$, it follows from 2.7 (b) and 2.7 (c) that $\mu_1 = \tau * \rho$. □

16.7. Let $X = \mathbb{T}^2$. Denote by $x = (z, w)$, $z, w \in \mathbb{T}$, elements of the group X. We have $Y \cong \mathbb{Z}^2$. To avoid introducing new notation we will assume that $Y = \mathbb{Z}^2$. Denote by $y = (m, n)$, $m, n \in \mathbb{Z}$, elements of the group Y. We recall that according to 1.14 (e) every automorphism $\delta \in \mathrm{Aut}(\mathbb{T}^2)$ is defined by an integer-valued matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $|ad - bc| = 1$. The automorphism δ acts on $\mathbb{T}^2$ as follows:

$$\delta(z, w) = (z^a w^c, z^b w^d), \quad (z, w) \in \mathbb{T}^2.$$

The adjoint automorphism $\varepsilon = \tilde{\delta} \in \mathrm{Aut}(\mathbb{Z}^2)$ is of the form

$$\varepsilon(m, n) = (am + bn, cm + dn), \quad (m, n) \in \mathbb{Z}^2.$$

We will identify the automorphisms δ and ε with the corresponding matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Let ξ_1 and ξ_2 be independent random variables with values in a group X and distributions μ_1 and μ_2. Assume that α_j, β_j, $j = 1, 2$, are topological automorphisms of X. It is easy to see that the study of possible distributions μ_j, provided that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, is reduced to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta \in \mathrm{Aut}(X)$.

Theorem 16.8. *Let $X = \mathbb{T}^2$. Assume that $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Aut}(X)$ satisfies the conditions $I \pm \delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Let G be the subgroup of X generated by all elements of order 2. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Then $\mu_j = \gamma_j * \rho_j$, where $\gamma_j \in \Gamma(X)$, $\sigma(\rho_j) \subset G$, $j = 1, 2$. Moreover, the distributions γ_j are concentrated on the cosets of a dense in X one-parameter subgroup of X.*

Proof. Set $\varepsilon = \tilde{\delta}$. By Lemma 16.1 if the conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i) which takes the form

$$\hat{\mu}_1(u + v)\hat{\mu}_2(u + \varepsilon v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u - \varepsilon v), \quad u, v \in Y. \tag{16.24}$$

Put $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (16.24). Put $\varphi_j(y) = -\ln \hat{\nu}_j(y)$. We conclude from (16.24) that the functions $\varphi_j(y)$ satisfy the equation

$$\varphi_1(u + v) + \varphi_2(u + \varepsilon v) - \varphi_1(u - v) - \varphi_2(u - \varepsilon v) = 0, \quad u, v \in Y. \tag{16.25}$$

We will solve equation (16.25) by the finite difference method. Arguing as in the proof of Theorem 16.2 we arrive at equation (16.9) for the function $\varphi_1(y)$. Taking into account that $n = 2$, we get

$$\Delta_{2k}\Delta_h^2\varphi_1(u) = 0, \quad k, h, u \in Y. \tag{16.26}$$

We deduce from (16.26) that the function $\varphi_1(y)$ satisfies the equation

$$\Delta_h^3 \varphi_1(y) = 0$$

on the subgroup $H_0 = Y^{(2)} = \{(2m, 2n) : m, n \in \mathbb{Z}\}$. It means that the restriction of the function $\varphi_1(y)$ to the subgroup H_0 is a polynomial of degree ≤ 2. Note that $\varphi_1(-y) = \varphi_1(y)$, $\varphi_1(y) \geq 0$, $y \in Y$, and $\varphi_1(0) = 0$. It follows from Lemma 10.10 that the function $\varphi_1(y)$ satisfies equation 2.16 (ii). This implies the representation

$$\varphi_1(y) = \langle Ay, y\rangle, \quad y \in H_0, \tag{16.27}$$

where A is a symmetric positive semidefinite matrix. The analogous representation we also obtain for the function $\varphi_2(y)$, i.e., $\varphi_2(y) = \langle By, y\rangle$, $y \in H_0$.

We represent the group Y as the union of cosets $Y = H_0 \cup H_1 \cup H_2 \cup H_3$, where $H_1 = (1,0) + H_0$, $H_2 = (0,1) + H_0$, $H_3 = (1,1) + H_0$. Set $\varphi_1^{(1)}(y) = \varphi_1((1,0) + y)$, $y \in Y$. Evidently, the function $\varphi_1^{(1)}(y)$ also satisfies equation (16.26). Hence the restriction of the function $\varphi_1^{(1)}(y)$ to the subgroup H_0 is a polynomial of degree ≤ 2. Since $\varphi_1(-y) = \varphi_1(y)$, $y \in H_1$, it is not difficult to verify that we get the representation

$$\varphi_1(u) = \langle A_1 u, u\rangle + r_1, \quad u \in H_1, \tag{16.28}$$

where A_1 is a symmetric matrix and $r_1 \in \mathbb{R}$.

Set $k = h$ in (16.26) and rewrite the obtained equation in the form

$$\varphi_1(u + 4h) - 2\varphi_1(u + 3h) + 2\varphi_1(u + h) - \varphi_1(u) = 0, \quad u, h \in Y.$$

Substituting here $u \in H_0$, $h \in H_1$ and taking into account (16.27) and (16.28), we find that

$$2\langle (A - A_1)h, h\rangle + \langle (A - A_1)u, h\rangle = 0, \quad u \in H_0,\ h \in H_1.$$

We conclude from this that $A = A_1$. Therefore, $\varphi_1(y) = \langle Ay, y\rangle + r_1$, $y \in H_1$. Arguing as above we obtain a representation for the function $\varphi_1(y)$ on the cosets H_2 and H_3. Finally we get

$$\varphi_1(y) = \langle Ay, y\rangle + r_j, \quad y \in H_j,\ j = 0, 1, 2, 3, \tag{16.29}$$

where $r_0 = 0$.

Arguing similarly we obtain an analogous representation for the function $\varphi_2(y)$,

$$\varphi_2(y) = \langle By, y\rangle + r'_j, \quad y \in H_j,\ j = 0, 1, 2, 3, \tag{16.30}$$

where $r'_0 = 0$. Substituting (16.29) and (16.30) into (16.25) and taking $u, v \in H_0$ we get that the matrices A and B satisfy the equation

$$A + B\varepsilon = 0.$$

Applying Lemma 16.5 we find

$$A = \sigma \begin{pmatrix} t_0^2 & t_0 \\ t_0 & 1 \end{pmatrix}, \quad B = kA,$$

where $\sigma \geq 0$, $t_0 = \frac{a-d-\sqrt{5}}{2b}$, $k = \frac{\sqrt{5}+a+d}{2}$. This yields the representation

$$\langle Ay, y\rangle = \langle A(m,n), (m,n)\rangle = \sigma(t_0 m + n)^2, \quad y = (m,n) \in Y. \tag{16.31}$$

It follows from (16.29) and (16.31) that

$$\begin{aligned} \hat{\nu}_1(y) &= \hat{\nu}_1(m,n) \\ &= \exp\{-\sigma(t_0 m + n)^2 - r_j\}, \quad y = (m,n) \in H_j,\ j = 0,1,2,3. \end{aligned} \tag{16.32}$$

Consider on the group Y the function

$$g(y) = g(m,n) = \exp\{-r_j\}, \quad y = (m,n) \in H_j,\ j = 0,1,2,3.$$

The function $g(y)$ takes constant values on each coset H_j. This implies that there exists a signed measure ω on the group X supported in $A(X, H_0)$ such that

$$\hat{\omega}(y) = g(y), \quad y \in Y. \tag{16.33}$$

Since $H_0 = Y^{(2)}$, by Theorem 1.9.5, $A(X, H_0) = X_{(2)} = G$. It is obvious that $G \cong (\mathbb{Z}(2))^2$. Denote by γ a Gaussian distribution on the group X with the characteristic function

$$\hat{\gamma}(y) = \hat{\gamma}(m,n) = \exp\{-\sigma(t_0 m + n)^2\}, \quad y = (m,n) \in Y. \tag{16.34}$$

We conclude from (16.32)–(16.34) that

$$\hat{\nu}_1(y) = \hat{\gamma}(y)\hat{\omega}(y), \quad y \in Y.$$

Then 2.7 (b) and 2.7 (c) imply that $\nu_1 = \gamma * \omega$. We will verify that the signed measure ω actually is a distribution.

Consider a Gaussian distribution λ on the group $\mathbb{R}$ with the characteristic function $\hat{\lambda}(s) = \exp\{-\sigma s^2\}$. Denote by $\tau\colon Y \mapsto \mathbb{R}$ the homomorphism $\tau(m,n) = t_0 m + n$. Let $p = \tilde{\tau}\colon \mathbb{R} \mapsto X$ be the adjoint homomorphism. It follows from Proposition 2.10 and 2.7 (b) that $\gamma = p(\lambda)$. Since t_0 is an irrational number it follows that the subgroup $\tau(Y)$ is dense in $\mathbb{R}$. By 1.13 (b) p is a monomorphism. Moreover, taking into account that τ is a monomorphism, we conclude from 1.13 (b) that the image $p(\mathbb{R})$ is dense in X. Denote by ζ_j, $j = 0,1,2,3$, the elements of the group G. Then

$$\omega = \sum_{j=0}^{3} a_j E_{\zeta_j},$$

where $a_j \in \mathbb{R}$. We have

$$\nu_1 = \gamma * \omega = \sum_{j=0}^{3} a_j(\gamma * E_{\zeta_j}). \tag{16.35}$$

The distribution $\gamma * E_{\zeta_j}$ is concentrated on the Borel set $p(\mathbb{R})\zeta_j$, and these sets are mutually disjoint. Since $\nu_1 \in \mathrm{M}^1(X)$, we deduce from (16.35) that all $a_j \geq 0$, i.e., $\omega \in \mathrm{M}^1(X)$.

Denote by ι the natural embedding $\iota\colon G \mapsto X$. Obviously, $p(\mathbb{R}) \cap G = \{0\}$. Extend p to the monomorphism $\hat{p}\colon \mathbb{R} \times G \mapsto X$ as follows: $\hat{p}(t, \zeta_j) = p(t)\iota(\zeta_j) = p(t)\zeta_j$, $(t, \zeta_j) \in \mathbb{R} \times G$. By Corollary 2.5 the monomorphism $\hat{p}$ generates an isomorphism of the semigroups $\mathrm{M}^1(\mathbb{R} \times G)$ and $\mathrm{M}^1(p(\mathbb{R}) \times G)$. Let $\upsilon = \lambda \otimes \omega \in \mathrm{M}^1(\mathbb{R} \times G)$ be the direct product of the distributions $\lambda \in \mathrm{M}^1(\mathbb{R})$ and $\omega \in \mathrm{M}^1(G)$. If we consider the distributions λ and ω as distributions on the group $\mathbb{R} \times G$, we have $\upsilon = \lambda * \omega$. Hence $\hat{p}(\upsilon) = p(\lambda) * \iota(\omega) = \gamma * \omega = \nu_1$. Obviously, the distribution ν_1 is concentrated on the Borel subgroup $p(\mathbb{R}) \times G$ of X. Taking into account that μ_1 is a factor of ν_1, by Proposition 2.2 we can substitute the distribution μ_1 by its shift μ_1' in such a way that the distribution μ_1' is also concentrated on the subgroup $p(\mathbb{R}) \times G$ and $\nu_1 = \mu_1' * \bar{\mu}_1'$. It follows from this that $\upsilon = \hat{p}^{-1}(\nu_1) = \hat{p}^{-1}(\mu_1') * \hat{p}^{-1}(\bar{\mu}_1')$. Set $\upsilon_1 = \hat{p}^{-1}(\mu_1)$. Then $\upsilon_1 \in \mathrm{M}^1(\mathbb{R} \times G)$. By Lemma 16.6 the distribution υ_1 has the form $\upsilon_1 = \tau * \rho$, where $\tau \in \Gamma(\mathbb{R})$, $\rho \in \mathrm{M}^1(G)$, because υ_1 is a factor of υ. We conclude from this that $\mu_1' = \hat{p}(\upsilon_1) = p(\tau) * \iota(\rho) = \gamma_1 * \rho$, where $\gamma_1 = p(\tau) \in \Gamma(X)$. This implies that the distribution μ_1 also has the desired form. Arguing as above we also get the required representation for the distribution μ_2. □

Remark 16.9. It should be noted that on the two-dimensional torus $X = \mathbb{T}^2$, three automorphisms $\delta_j \in \mathrm{Aut}(X)$, $j = 1, 2, 3$, such that $\delta_i \pm \delta_j \in \mathrm{Aut}(X)$ for all $i \neq j$ do not exist. Indeed, without loss of generality we can assume that $\delta_1 = I$. Let $\delta_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\delta_3 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. As has been noted in the proof of Lemma 16.5, if $|\det \delta_j| = |\det(I \pm \delta_j)| = 1$ for $j = 2, 3$, then $ad - bc = -1$ and $a'd' - b'c' = -1$. On the other hand, since $\delta_2 \pm \delta_3 \in \mathrm{Aut}(X)$, we have $|\det(\delta_2 \pm \delta_3)| = 1$. It follows from this that $|2 + k| = |2 - k| = 1$, where $k = ad' + da' - bc' - cb'$, but it is impossible. Therefore we can formulate Theorem 16.8 as a statement of $n \geq 2$ independent random variables ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, taking values in the two-dimensional torus $X = \mathbb{T}^2$, where automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy the conditions $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. It follows from the above remark that then $n = 2$.

Remark 16.10. Let $X = \mathbb{T}^2$. Theorem 16.8 is false if the characteristic functions of the considered distributions can vanish. To construct an example take an automorphism $\delta \in \mathrm{Aut}(X)$ of the form

$$\delta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $I \pm \delta \in \mathrm{Aut}(X)$. Set $K = X_{(4)}$.

Then $K \cong (\mathbb{Z}(4))^2$. Denote by $(e^{\frac{\pi i m}{2}}, e^{\frac{\pi i n}{2}})$, $m, n \in \{0, 1, 2, 3\}$, elements of the group K. The automorphism δ acts on K as follows:

$$\delta(e^{\frac{\pi i m}{2}}, e^{\frac{\pi i n}{2}}) = (e^{\frac{\pi i (m+n)}{2}}, e^{\frac{\pi i m}{2}}).$$

Consider the following subgroups of K:

$$K_1 = \{(1, 1), (1, -1)\}, \quad K_2 = \{(1, 1), (-1, -1)\},$$
$$F = \{(1, 1), (1, -1), (-1, 1), (-1, -1), (i, 1), (-i, 1), (i, -1), (-i, -1)\}.$$

Let ξ_1 and ξ_2 be independent random variables with values in the group X and distributions $\mu_1 = am_{K_1} + (1-a)m_F$ and $\mu_2 = am_{K_2} + (1-a)m_F$, where $0 < a < 1$. Set $\varepsilon = \tilde{\delta}$. As will be proved below (Lemma 17.11), the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24). By Lemma 16.1 the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. On the other hand, obviously, the distributions μ_j can not be represented in the form $\mu_j = \gamma_j * \rho_j$, where $\gamma_j \in \Gamma(X)$, $\rho_j \in \mathrm{M}^1(G)$.

17 Random variables with values in finite and discrete Abelian groups

In this section we continue to study group analogues of the Heyde theorem. Let X be a countable discrete Abelian group. Then by Proposition 3.6, $\Gamma(X) = D(X)$ and a natural analogue of the class of Gaussian distributions for such groups is the class of idempotent distributions. Let Y be the character group of the group X, α_j, β_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be automorphisms of X satisfying the conditions $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables with values in X and distributions μ_j. Our main attention will be devoted to the following problem: for which groups X does it follow from the symmetry of the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ that all μ_j are idempotent distributions? In contrast to Section 16 we do not assume that the characteristic functions $\hat{\mu}_j(y)$ do not vanish.

First consider the case of a finite group X and two random variables.

Theorem 17.1. *Let X be a finite group containing no elements of order 2. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in$ $\mathrm{Aut}(X)$. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_1, \mu_2 \in I(X)$.*

Proof. Passing to the new random variables $\zeta_j = \alpha_j\xi_j$ we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, $\delta_j \in \mathrm{Aut}(X)$. The condition $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$ is transformed into the condition $\delta_1 \pm \delta_2 \in \mathrm{Aut}(X)$. It is clear that we can assume that $L_2 = \xi_1 + \delta\xi_2$, where $\delta,\ I \pm \delta \in \mathrm{Aut}(X)$. Set $\varepsilon = \tilde{\delta}$. By Lemma 16.1 the symmetry of the conditional distribution of L_2 given L_1 implies that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i) which takes the form (16.24). Put $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$ and rewrite equation (16.24) using this notation. We get

$$f(u+v)g(u+\varepsilon v) = f(u-v)g(u-\varepsilon v), \quad u, v \in Y. \tag{17.1}$$

Set $\nu_j = \mu_j * \bar{\mu}_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. It is obvious that the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (17.1). If we prove that $\nu_j \in I(X)$, then by 2.7 (b) and 2.7 (e), $\mu_j \in I(X)$. Thus we can solve equation (17.1) assuming that $f(y) \geq 0$, $g(y) \geq 0$, $f(-y) = f(y)$, $g(-y) = g(y)$. We will prove that in this case $f(y) = g(y) = \widehat{m}_K(y)$, where K is a subgroup of X. The statement of the theorem follows from this.

Put $\alpha = I + \varepsilon$, $\beta = I - \varepsilon$. We conclude from the condition of the theorem that $\alpha, \beta \in \mathrm{Aut}(Y)$. Set $\kappa = \beta\alpha^{-1}$.

Substituting $v = -u$ into (17.1) we get

$$g(\beta u) = f(2u)g(\alpha u), \quad u \in Y.$$

Replace here u by $\alpha^{-1}u$. We obtain

$$g(\kappa u) = f(2\alpha^{-1}u)g(u), \quad u \in Y. \tag{17.2}$$

Since $0 \leq f(y) \leq 1$, $0 \leq g(y) \leq 1$, (17.2) yields that

$$g(\kappa u) \leq g(u), \quad u \in Y.$$

Note that $Y \cong X$, hence Y is a finite group. Therefore, its automorphism group $\mathrm{Aut}(Y)$ is also finite. Hence $\kappa^n = I$ for some natural n. We will assume that n is the smallest one. We have

$$g(y) = g(\kappa^n y) \leq \cdots \leq g(\kappa y) \leq g(y), \quad y \in Y.$$

It follows from this that

$$g(y) = g(\kappa y) = \cdots = g(\kappa^{n-1} y), \quad y \in Y. \tag{17.3}$$

Substituting $u = -\varepsilon v$ into (17.1) and taking into account that $f(-y) = f(y)$, $g(-y) = g(y)$, we get

$$f(\beta v) = f(\alpha v)g(2\varepsilon v), \quad v \in Y.$$

Replace here v by $\alpha^{-1}v$. We obtain

$$f(\kappa v) = f(v)g(2\varepsilon\alpha^{-1}v), \quad v \in Y. \tag{17.4}$$

Arguing similarly we arrive at

$$f(y) = f(\kappa y) = \cdots = f(\kappa^{n-1} y), \quad y \in Y. \tag{17.5}$$

Thus the functions $f(y)$ and $g(y)$ take constant values on each of the orbits $O_y = \{y, \kappa y, \ldots, \kappa^{n-1} y\}$. This value generally depends on y. We note that in proving (17.3) and (17.5) we did not use that the group X contains no elements of order 2.

Set

$$N_f = \{y \in Y : f(y) \neq 0\}, \quad E_f = \{y \in Y : f(y) = 1\},$$

and

$$N_g = \{y \in Y : f(y) \neq 0\}, \quad E_g = \{y \in Y : f(y) = 1\}.$$

Since $X \cong Y$ and $X_{(2)} = \{0\}$, we have $Y_{(2)} = \{0\}$. Hence $f_2 \in \mathrm{Aut}(Y)$. We recall that for a finite set F we denote by $|F|$ the number of elements of F. We conclude from (17.2) and (17.3) that if $y \in N_g$, then

$$f(2\alpha^{-1} y) = 1. \tag{17.6}$$

Hence (17.6) implies that $|N_g| \leq |E_f|$. Arguing similarly we find from (17.4) and (17.5) that if $y \in N_f$, then $g(2\varepsilon\alpha^{-1} y) = 1$. This implies the inequality $|N_f| \leq |E_g|$. So, we get

$$|N_g| \leq |E_f| \leq |N_f|, \quad |N_f| \leq |E_g| \leq |N_g|.$$

It follows from this that

$$|N_f| = |E_f|, \quad |N_g| = |E_g|, \tag{17.7}$$

and (17.7) implies that

$$N_f = E_f, \quad N_g = E_g. \tag{17.8}$$

Since $f_2 \in \mathrm{Aut}(Y)$, we have

$$I + \kappa = f_2 \alpha^{-1} \in \mathrm{Aut}(Y). \tag{17.9}$$

We deduce from (17.3) and (17.8) that $\kappa(E_g) = E_g$. Whence, in view of (17.9) we find

$$\alpha(E_g) = E_g. \tag{17.10}$$

We conclude from (17.6) that $N_g = E_g \subset E_f$. Therefore $E_f = E_g = E$. So, we have proved that

$$f(y) = g(y) = \begin{cases} 1 & \text{if } y \in E, \\ 0 & \text{if } y \notin E. \end{cases}$$

Put $K = A(X, E)$. Then by Theorem 1.9.1, $E = A(Y, K)$ and it follows from 2.14 (i) and 2.7 (b) that $\mu_1 = \mu_2 = m_K$. We also note that since $\varepsilon = \alpha - I$, (17.10) implies that $\varepsilon(E) = E$. □

Corollary 17.2. *Let X be a finite group containing no elements of order 2, δ be an automorphism of X such that $I \pm \delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in the group X and distributions μ_1 and μ_2. If the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = m_K * E_{x_j}$, where K is a subgroup of X, and $x_j \in X$, $j = 1, 2$. Moreover, $\tilde{\delta}(A(Y, K)) = A(Y, K)$.*

Corollary 17.3. *Let Y be a finite group, ε be an automorphism of Y such that $I \pm \varepsilon \in \mathrm{Aut}(Y)$. Put $\kappa = (I - \varepsilon)(I + \varepsilon)^{-1}$. If characteristic functions $f(y)$ and $g(y)$ satisfy equation (17.1), then $|f(y)| = |f(\kappa y)|$, $|g(y)| = |g(\kappa y)|$ for any $y \in Y$.*

Remark 17.4. The reasoning given in Remark 16.3 shows that Theorem 17.1 is false if a group X contains elements of order 2.

Remark 17.5. The condition

$$\beta_1\alpha_1^{-1} - \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X) \tag{17.11}$$

in Theorem 17.1 can be omitted. Indeed, without loss of generality we may assume that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, $\delta \in \mathrm{Aut}(X)$. Then condition (17.11) is transformed into the condition $I - \delta \in \mathrm{Aut}(X)$ which is equivalent to the condition $\beta \in \mathrm{Aut}(Y)$. Assume that $\beta \notin \mathrm{Aut}(Y)$. Since Y is a finite group, it means that $L = \mathrm{Ker}\,\beta \neq \{0\}$. Then if $u \in L$ we have $\varepsilon u = u$ and $\alpha u = 2u$. Substituting $u = v \in L$ into equation (17.1) we get

$$f(2u)g(2u) = 1, \quad u \in L. \tag{17.12}$$

Since $f_2 \in \mathrm{Aut}(Y)$, it follows from (17.12) that $f(y) = g(y) = 1$, $y \in L$. By Proposition 2.13 the functions $f(y)$ and $g(y)$ are L-invariant. Hence they induce functions $\tilde{f}([y])$ and $\tilde{g}([y])$ on the factor group Y/L, namely $\tilde{f}([y]) = f(y)$, $\tilde{g}([y]) = g(y)$, $y \in [y]$. The automorphism ε also induces an automorphism $\hat{\varepsilon}$ on the factor group Y/L by the formula $\hat{\varepsilon}[y] = [\varepsilon y]$, $y \in [y]$. So, we can consider equation (17.1) on the factor group Y/L. If the induced homomorphism $\hat{\beta} \notin \mathrm{Aut}(Y/L)$, we repeat this procedure until we obtain $\hat{\beta} \in \mathrm{Aut}(Y/L)$. Then we apply Theorem 17.1. After returning to the initial distributions we get $\mu_j \in I(X)$.

We supplement Theorem 17.1 with the following statement.

Proposition 17.6. *Let ξ_1 and ξ_2 be independent identically distributed random variables with values in a group X and distribution m_K, where K is a compact subgroup of X. Let $\delta \in \mathrm{Aut}(X)$ and $I \pm \delta \in \mathrm{Aut}(X)$. Then the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if*

(i) $\gamma(K) \supset K$,

where $\gamma = (I + \delta)^{-1}(I - \delta)$.

Proof. We retain the notation used in the proof of Theorem 17.1. Put $L = A(Y, K)$, $f(y) = \widehat{m}_K(y)$. We note that $\kappa = \tilde{\gamma}$. By Lemma 16.1 if the conditional distribution of L_2 given L_1 is symmetric, then the characteristic functions of the random variables ξ_j satisfy equation 16.1 (i) which takes the form

$$f(u + v)f(u + \varepsilon v) = f(u - v)f(u - \varepsilon v), \quad u, v \in Y. \tag{17.13}$$

Substituting $u = v$ into (17.13) we get $f(2u)f(\alpha u) = f(\beta u)$, $u \in Y$. It follows from this that

$$f(\kappa u) = f(2\alpha^{-1}u)f(u), \quad u \in Y. \tag{17.14}$$

Therefore, if $\kappa y \in L$, then $y \in L$. Hence by Lemma 13.10, $\gamma(K) \supset K$, i.e., (i) holds.

Assume that (i) is fulfilled. We will show that the characteristic function $f(y)$ satisfies equation (17.13). Suppose that for some $u, v \in Y$ the left-hand side of equation (17.13) is equal to 1. This implies that

$$u + v \in L, \quad u + \varepsilon v \in L, \tag{17.15}$$

and therefore $\beta v = (I - \varepsilon)v \in L$. Since $\beta v = \kappa\alpha v$, by Lemma 13.10 we have that

$$\alpha v = (I + \varepsilon)v \in L. \tag{17.16}$$

We conclude from (17.15) and (17.16) that $u - v \in L, u - \varepsilon v \in L$. Hence the right-hand side of equation (17.13) is also equal to 1. Arguing similarly we verify that if for some $u, v \in Y$ the right-hand side of equation (17.13) is equal to 1, then the left-hand side of equation (17.13) is equal to 1. Thus we have proved that the characteristic function $f(y) = \widehat{m}_K(y)$ satisfies equation (17.13). Hence by Lemma 16.1 the conditional distribution of the linear form L_2 given L_1 is symmetric. □

Note that in the proof of Theorem 17.1 we assumed that there exist automorphisms α_j, β_j, $j = 1, 2$, of a group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. We will describe now the finite Abelian groups X which have this property. Then we will apply this result to prove a group analogue of the Heyde theorem for an arbitrary number of independent random variables. Clearly, it suffices to find out when there exists $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$.

Proposition 17.7. *Let X be a finite group,*

$$\text{(i)} \quad X = \mathop{\mathbf{P}}_{p \in \mathcal{P}} X_p$$

be the decomposition of X into a direct product of its p-components. Then the following statements are equivalent:

(ii) *there exists $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$;*

(iii) *for both $p = 2$ and $p = 3$ either $X_p = \{0\}$ or the decomposition of X_p into a direct product of cyclic subgroups contains each cyclic factor with multiplicity not less than 2.*

Proof. Note that each p-component X_p is a characteristic subgroup of X. It is clear that (ii) holds if and only if for each prime p such that $X_p \neq \{0\}$ there exists $\delta \in \mathrm{Aut}(X_p)$ such that

$$I \pm \delta \in \mathrm{Aut}(X_p). \tag{17.17}$$

If $p > 3$ we set $\delta = f_2$. Then $\delta,\ I \pm \delta \in \mathrm{Aut}(X_p)$. Thus we should find out when condition (17.17) holds for $p = 2$ and $p = 3$. In what follows we consider only the group X_3. For the group X_2 we argue similarly. Put $K = X_3$. By Theorem 1.19.4 we have

$$K \cong \mathbf{P}_j (\mathbb{Z}(3^{k_j}))^{n_j}, \quad k_j < k_{j+1}.$$

The numbers k_j and n_j are uniquely determined by the group K. To avoid introducing new notation we will suppose that

$$K = \mathbf{P}_j (\mathbb{Z}(3^{k_j}))^{n_j}, \quad k_j < k_{j+1}. \tag{17.18}$$

We will prove that there exists $\delta \in \mathrm{Aut}(K)$ such that $I \pm \delta \in \mathrm{Aut}(K)$ if and only if all $n_j \geq 2$ in (17.18). We note that for every natural n the subgroups $K_{(n)}$ and $K^{(n)}$ are characteristic. Hence the subgroups $H_j = K^{(3^{k_j - 1})} \cap K_{(3)}$ are also characteristic. Therefore any automorphism $\alpha \in \mathrm{Aut}(K)$ induces an automorphism $\hat{\alpha}$ on the factor group H_{j_0}/H_{j_0+1}. Assume that $n_{j_0} = 1$ for some j_0. Then as easily seen, $H_{j_0}/H_{j_0+1} \cong \mathbb{Z}(3)$. Hence if $\delta, I \pm \delta \in \mathrm{Aut}(K)$, then $\hat{\delta}, I \pm \hat{\delta} \in \mathrm{Aut}(H_{j_0}/H_{j_0+1})$. Taking into account that there are only two automorphisms $\pm I$ on the group $\mathbb{Z}(3)$, we obtain the contradiction. Thus we have proved that (ii) implies (iii).

Let us prove that (iii) implies (ii). Set $G_1 = (\mathbb{Z}(3^r))^2$. Let the automorphism $\delta_1 \in \mathrm{Aut}(G_1)$ be defined by $\delta_1(k,l) = (k + l, k)$, $k, l \in \mathbb{Z}(3^r)$. It is obvious that $I \pm \delta_1 \in \mathrm{Aut}(G_1)$. For the group $G_2 = (Z(3^r))^3$ we put $\delta_2(k,l,m) = (k + l + m, k + l, k)$, $k, l, m \in \mathbb{Z}(3^r)$. Then $\delta_2, I \pm \delta_2 \in \mathrm{Aut}(G_2)$. If in (17.18) all $n_j \geq 2$, then the group K is a finite direct product of groups each of which is isomorphic to either G_1 or G_2. The desired automorphism $\delta \in \mathrm{Aut}(X)$ can be constructed as a finite direct product of the automorphisms δ_1 and δ_2. □

Now assume that a finite group X contains elements of order 2 and describe distributions which are characterized by the symmetry of the conditional distribution of one linear form given another.

Theorem 17.8. *Let X be a finite group, X_2 be the 2-component of X. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$ be automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_j = \rho_j * \pi_j$, where $\sigma(\rho_j) \subset X_2$, $\pi_j \in I(X)$, $j = 1, 2$.*

Proof. Arguing as in the proof of Theorem 17.1, we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$. By Lemma 16.1 it follows from the symmetry of the conditional distribution of the linear form L_2 given L_1 that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i).

Decompose the group X into a direct product of its p-components:

$$X = \mathop{\mathbf{P}}_{p \in \mathcal{P}} X_p.$$

Set

$$G = X_2, \quad K = \mathop{\mathbf{P}}_{p>2} X_p.$$

Then $X = G \times K$. If $G = \{0\}$, then the assertion of the theorem follows from Theorem 17.1. Assume that $G \neq \{0\}$. By Theorem 1.7.1, $Y \cong H \times L$, where $H = G^*$, $L = K^*$. To avoid introducing new notation we will suppose that $Y = H \times L$. Denote by $y = (h, l)$, $h \in H$, $l \in L$, elements of the group Y. Since the subgroups H and L are characteristic, any automorphism $\tau \in \mathrm{Aut}(Y)$ can be written in the form $\tau(h, l) = (\tau h, \tau l)$, $(h, l) \in Y$.

Put $\varepsilon = \tilde{\delta}$, $\alpha = I + \varepsilon$, $\beta = I - \varepsilon$, $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$ and rewrite equation 16.1 (i) in the form:

$$\begin{aligned} &f(h + h', l + l')g(h + \varepsilon h', l + \varepsilon l') \\ &\quad = f(h - h', l - l')g(h - \varepsilon h', l - \varepsilon l'), \quad (h, l), (h', l') \in Y. \end{aligned} \tag{17.19}$$

Substituting $h = h' = 0$ into (17.19) we get

$$f(0, l + l')g(0, l + \varepsilon l') = f(0, l - l')g(0, l - \varepsilon l'), \quad l, l' \in L. \tag{17.20}$$

Since the group K contains no elements of order 2, by Corollary 17.2 any solution of equation (17.20) has the form

$$f(0, l) = (k_1, l)\widehat{m}_F(l), \quad g(0, l) = (k_2, l)\widehat{m}_F(l), \quad l \in L, \tag{17.21}$$

where F is a subgroup of the group K, and $k_j \in K$. Put $E = A(L, F)$. Substituting (17.21) into (17.20) we get $2(k_1 + \delta k_2) \in F$. In view of $K_{(2)} = \{0\}$ we have $k_1 + \delta k_2 \in F$. Set $k = k_1 + \delta k_2$. It is clear that representation (17.21) does not change if we substitute $k_1' = k_1 - k$ for k_1 in (17.21). But then $k_1' + \delta k_2 = 0$. It is easy to see that in this case the characteristic functions $(-k_1', l)$ and $(-k_2, l)$ satisfy equation (17.20). Consider the distributions $\mu_1' = \mu_1 * E_{-k_1'}$ and $\mu_2' = \mu_2 * E_{-k_2}$. Set $f'(y) = \hat{\mu}_1'(y)$, $g'(y) = \hat{\mu}_2'(y)$. We conclude from 2.7 (c) that the characteristic functions $f'(y)$ and $g'(y)$ also satisfy equation (17.19).

Put $B = L/E$. We have $Y/E \cong H \times B$. By Theorem 1.9.2, $F^* \cong B$ and hence $(G \times F)^* \cong Y/E$. Taking into account 2.14 (i) and 2.7 (c), it follows from (17.21) that

$$f'(0, h) = g'(0, h) = \begin{cases} 1 & \text{if } l \in E, \\ 0 & \text{if } l \notin E. \end{cases} \tag{17.22}$$

By Proposition 2.13, (17.22) implies that the characteristic functions $f'(y)$ and $g'(y)$ are E-invariant. Obviously, K is a characteristic subgroup. Consider independent random variables ζ_1 and ζ_2 with values in the group K and distributions $m_F * E_{k_1}$ and

$m_F * E_{k_2}$. By Lemma 16.1 we deduce from (17.20) that the conditional distribution of the linear form $L_2 = \zeta_1 + \delta\zeta_2$ given $L_1 = \zeta_1 + \zeta_2$ is symmetric. Note that $f_2 \in \mathrm{Aut}(K)$ and apply Corollary 17.2 to the group K. We get $\varepsilon(E) = E$. Hence we can pass from equation (17.19) for the functions $f'(y)$ and $g'(y)$ on the group Y to the induced equation on the factor group Y/E putting $\tilde{f}([y]) = f'(y)$, $\tilde{g}([y]) = g'(y)$, $\hat{\varepsilon}[y] = [\varepsilon y]$. It follows from $\varepsilon(E) = E$ that $\hat{\varepsilon} \in \mathrm{Aut}(Y/E)$. It means that we pass from consideration of the random variables with values in the group $X = G \times K$ to random variables taking values in the subgroup $G \times F$. Obviously, the solutions of the induced equation satisfy the condition

$$\{b \in B : \tilde{f}(0, b) = 1\} = \{b \in B : \tilde{g}(0, b) = 1\} = \{0\}.$$

Thus we have

$$\tilde{f}(0, b) = \tilde{g}(0, b) = \begin{cases} 1 & \text{if } b = 0, \\ 0 & \text{if } b \neq 0. \end{cases} \tag{17.23}$$

Consider equation (17.19) for the characteristic functions $\tilde{f}(s, b)$ and $\tilde{g}(s, b)$ on the factor group $Y/E \cong H \times B$. We get

$$\tilde{f}(h+h', b+b')\tilde{g}(h+\hat{\varepsilon}h', b+\hat{\varepsilon}b') = \tilde{f}(h-h', b-b')\tilde{g}(h-\hat{\varepsilon}h', b-\hat{\varepsilon}b'), \tag{17.24}$$

$(h, b), (h', b') \in H \times B$. Note that in view of $\varepsilon(E) = E$ we have $\alpha(E) \subset E$. Since $\alpha \in \mathrm{Aut}(Y)$ and E is a finite group, we have $\alpha(E) = E$, whence $\hat{\alpha} \in \mathrm{Aut}(Y/E)$. Take $u \in H$, $v \in B$ and substitute $h = \hat{\alpha}^{-1}u$, $h' = -\hat{\alpha}^{-1}u$, $b = \hat{\alpha}^{-1}v$, $b' = -\hat{\alpha}^{-1}v$ into (17.24). We obtain

$$\tilde{g}(\kappa u, \kappa v) = \tilde{f}(2\hat{\alpha}^{-1}u, 2\hat{\alpha}^{-1}v)\tilde{g}(u, v), \tag{17.25}$$

where $\kappa = \hat{\beta}\hat{\alpha}^{-1}$. Assume that there exists an element $(h_0, b_0) \in H \times B$, $b_0 \neq 0$ such that $\tilde{g}(h_0, b_0) \neq 0$. Then by Corollary 17.3, $|\tilde{g}(\kappa h_0, \kappa b_0)| = |\tilde{g}(h_0, b_0)| \neq 0$, and (17.25) yields that

$$|\tilde{f}(2\hat{\alpha}^{-1}h_0, 2\hat{\alpha}^{-1}b_0)| = 1. \tag{17.26}$$

It follows from Proposition 2.13 that the subset of Y where the module of a characteristic function is equal to 1 is a subgroup. Taking this into account, (17.26) implies that for any natural k the equality $|\tilde{f}(2^k\hat{\alpha}^{-1}h_0, 2^k\hat{\alpha}^{-1}b_0)| = 1$ is fulfilled. Since G is a 2-primary group, H is also a 2-primary group. Note that if h_0 is an element of order 2^m, then $2^m\hat{\alpha}^{-1}h_0 = 0$. Hence

$$|\tilde{f}(0, 2^m\hat{\alpha}^{-1}b_0)| = 1.$$

On the other hand, since the group B contains no elements of order 2 and $\hat{\alpha}^{-1} \in \mathrm{Aut}(H \times B)$, we have $2^m\alpha^{-1}b_0 \neq 0$, contrary to (17.23). Thus we have proved that $\tilde{g}(h, b) = 0$, if $b \neq 0$. Hence the characteristic function $\tilde{g}(h, b)$ is represented in the form

$$\tilde{g}(h, b) = \begin{cases} g_0(h) & \text{if } b = 0, \\ 0 & \text{if } b \neq 0, \end{cases}$$

where $g_0(h) = \tilde{g}(h,0)$. We get $\tilde{g}(h,b) = g_0(h)\widehat{m}_F(b)$, $(h,b) \in H \times B$. The function $g_0(h)$ is the characteristic function of a distribution ρ_2 such that $\sigma(\rho_2) \subset G$. We conclude from 2.7 (b) and 2.7 (c) that $\mu_2' = \rho_2 * m_F$. This implies that $\mu_2 = \rho_2 * \pi_2 * E_{k_2}$. For the distribution μ_1 we argue similarly. □

Corollary 17.9. *Let X be a finite group. Represent X in the form $X = X_2 \times K$, where $K = \mathbf{P}_{p>2}\, X_p$, X_p is the p-component of X. Let δ be an automorphism of the group X such that $I \pm \delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. If the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \rho_j * m_F * E_{x_j}$, where $\sigma(\rho_j) \subset X_2$, F is a subgroup of K, $x_j \in K$, $j = 1, 2$. Moreover, the characteristic functions of the distributions $\mu_j' = \mu_j * E_{-x_j}$ also satisfy equation (16.24).*

17.10. Let X be a finite 2-primary group, i.e., $X = X_2$ in decomposition 17.7 (i). Let δ be an automorphism of X such that $I \pm \delta \in \mathrm{Aut}(X)$. Put $\varepsilon = \tilde{\delta}$. We will discuss the following problem. What are possible distributions μ_1 and μ_2 of independent random variables ξ_1 and ξ_2 taking values in the group X assuming that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric? Taking into account Lemma 16.1 we can reformulate this problem as follows: what are distributions μ_j such that their characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24)?

The corresponding idempotent distributions μ_j such that $\mu_1 = \mu_2 = m_K$ were described in Proposition 17.6. Moreover, as has been noted in Remark 16.3, if μ_j are arbitrary distributions such that $\sigma(\mu_j) \subset X_{(2)}$, then μ_j are also solutions of the problem. Under the assumption that $X_2 \neq X_{(2)}$ we mention one more class of distributions μ_j such that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24). They are distributions μ_j which are invariant with respect to the subgroup $X^{(2)}$. Let us verify this. In view of Theorem 1.9.5 it is easily seen that the characteristic functions $\hat{\mu}_j(y)$ of such distributions are of the form:

$$\hat{\mu}_1(y) = \begin{cases} f_0(y) & \text{if } y \in Y_{(2)}, \\ 0 & \text{if } y \notin Y_{(2)}, \end{cases} \qquad \hat{\mu}_2(y) = \begin{cases} g_0(y) & \text{if } y \in Y_{(2)}, \\ 0 & \text{if } y \notin Y_{(2)}, \end{cases}$$

where $f_0(y)$ and $g_0(y)$ are arbitrary characteristic functions on the subgroup $Y_{(2)}$. It is easy to make sure that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24). Indeed, let $u, v \in Y_{(2)}$. Then $u + v = u - v$, $u + \varepsilon v = u - \varepsilon v$. Hence $\hat{\mu}_1(u + v) = \hat{\mu}_1(u - v)$, $\hat{\mu}_2(u + \varepsilon v) = \hat{\mu}_2(u - \varepsilon v)$, and equation (16.24) is satisfied. Let either $u \in Y_{(2)}, v \notin Y_{(2)}$ or $u \notin Y_{(2)}, v \in Y_{(2)}$. Then $u \pm v \notin Y_{(2)}$, hence $\hat{\mu}_1(u \pm v) = 0$, and both sides of equation (16.24) are equal to zero. If $u, v \notin Y_{(2)}$, then the left-hand side of equation (16.24) is equal to zero, because in the opposite case we have $u + v \in Y_{(2)}$, $u + \varepsilon v \in Y_{(2)}$. This implies that $(I - \varepsilon)v \in Y_{(2)}$, and hence $v \in Y_{(2)}$, contrary to the assumption. Arguing as above we verify that the right-hand side of equation (16.24) is also equal to zero for $u, v \notin Y_{(2)}$. Thus the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24).

Thus if $X_2 \neq X_{(2)}$, then there exist three different classes of distributions μ_j such that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (16.24):

(I) idempotent distributions;
(II) distributions supported in $X_{(2)}$;
(III) distributions invariant under shifts by elements of $X^{(2)}$.

Obviously, if the characteristic functions of distributions μ_1 and μ_2 satisfy equation (16.24), and the same is true for distributions ν_1 and ν_2, then in view of 2.7 (c) the distributions $\gamma_1 = \mu_1 * \nu_1$ and $\gamma_2 = \mu_2 * \nu_2$ also have this property. It turns out that if $X_2 \neq X_{(2)}$, then there exist distributions μ_1, μ_2 on X such that they do not belong to the classes (I)–(III) and they are not convolutions of distributions of these classes, whereas their characteristic functions satisfy equation (16.24). This situation is opposite to the case when $X_2 = \{0\}$, because if $X_2 = \{0\}$ only the characteristic functions of idempotent distributions satisfy equation (16.24) (see Theorem 17.1). We need the following

Lemma 17.11. *For each of the groups $X = (\mathbb{Z}(4))^2$ and $X = (\mathbb{Z}(4))^3$ there exist an automorphism $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$ and independent random variables ξ_1 and ξ_2 with values in X and distributions μ_1 and μ_2 such that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric and μ_j can not be represented as convolutions of distributions of the classes* (I)–(III) *defined in* 17.10.

Proof. Let $X = (\mathbb{Z}(4))^2$. Then $Y \cong (\mathbb{Z}(4))^2$. Denote by $x = (m, n)$ elements of the group X and by $y = (k, l)$ elements of the group Y, where $m, n, k, l \in \mathbb{Z}(4)$. Let an automorphism $\delta \in \mathrm{Aut}(X)$ be of the form $\delta(m, n) = (m + n, m)$. Put $\varepsilon = \tilde{\delta}$. Then $\varepsilon(k, l) = (k + l, k)$. It is obvious that $I \pm \delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions

$$\mu_1 = am_{K_1} + (1 - a)m_F, \quad \mu_2 = am_{K_2} + (1 - a)m_F,$$

where $K_1 = \{(0,0), (0,2)\}$, $K_2 = \{(0,0), (2,2)\}$, $F = \{X_{(2)}, (1,0) + X_{(2)}\}$, and $0 < a < 1$. It is easy to compute that

$$A(Y, K_1) = \{Y_{(2)}, (1,0) + Y_{(2)}\},$$
$$A(Y, K_2) = \{Y_{(2)}, (1,1) + Y_{(2)}\},$$
$$A(Y, F) = \{(0,0), (0,2)\},$$

and hence the characteristic functions $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$ are of the form

$$f(y) = \begin{cases} 1 & \text{if } y \in \{(0,0), (0,2)\}, \\ a & \text{if } y \in \{(2,0), (2,2), (1,0) + Y_{(2)}\}, \\ 0 & \text{if } y \in \{(1,1) + Y_{(2)}, (0,1) + Y_{(2)}\}, \end{cases}$$

$$g(y) = \begin{cases} 1 & \text{if } y \in \{(0,0), (0,2)\}, \\ a & \text{if } y \in \{(2,0), (2,2), (1,1) + Y_{(2)}\}, \\ 0 & \text{if } y \in \{(1,0) + Y_{(2)}, (0,1) + Y_{(2)}\}. \end{cases}$$

We will verify that the characteristic functions $f(y)$ and $g(y)$ satisfy equation (17.1). Obviously, for arbitrary $u, v \in Y$ we have either $u \pm v \in Y_{(2)}$ or $u \pm v \notin Y_{(2)}$. Therefore, the following cases are possible:

1. $u \pm v, u \pm \varepsilon v \in Y_{(2)}$. It follows from this that $(I - \varepsilon)v \in Y_{(2)}$, and hence $u, v \in Y_{(2)}$. Then $u + v = u - v$, $u + \varepsilon v = u - \varepsilon v$, and equation (17.1) holds.

2. $u \pm v \in Y_{(2)}$, $u \pm \varepsilon v \notin Y_{(2)}$. Since $u \pm v \in Y_{(2)}$, the following cases are possible:

A. $u, v \in (1,0) + Y_{(2)}$. Then $\varepsilon v \in (1,1) + Y_{(2)}$ and $g(u \pm \varepsilon v) = 0$. Hence both sides of equation (17.1) are equal to zero.

B. $u, v \in (1,1) + Y_{(2)}$. Then $\varepsilon v \in (0,1) + Y_{(2)}$ and $g(u \pm \varepsilon v) = 0$. Hence both sides of equation (17.1) are equal to zero.

C. $u, v \in (0,1) + Y_{(2)}$. Then $\varepsilon v \in (1,0) + Y_{(2)}$ and $g(u + \varepsilon v) = g(u - \varepsilon v) = a \neq 0$. Hence the left-hand side of equation (17.1) is not equal to the right-hand side if either $u + v \in \{(0,0), (0,2)\}$ and $u - v \in \{(2,0), (2,2)\}$ or $u - v \in \{(0,0), (0,2)\}$ and $u + v \in \{(2,0), (2,2)\}$. In each of these cases we have $2u \in \{(2,0), (2,2)\}$, and hence $u \in \{(1,0) + Y_{(2)}, (1,1) + Y_{(2)}\}$, contrary to the assumption. Thus both sides of equation (17.1) are equal.

3. $u \pm v \notin Y_{(2)}$, $u \pm \varepsilon v \in Y_{(2)}$. Since $u \pm \varepsilon v \in Y_{(2)}$, the following cases are possible:

A. $u \in (1,0) + Y_{(2)}$, $v \in (0,1) + Y_{(2)}$. Then $f(u \pm v) = 0$. Hence both sides of equation (17.1) are equal to zero.

B. $u \in (1,1) + Y_{(2)}$, $v \in (1,0) + Y_{(2)}$. Then $f(u \pm v) = 0$. Hence both sides of equation (17.1) are equal to zero.

C. $u \in (0,1) + Y_{(2)}$, $v \in (1,1) + Y_{(2)}$. Then $f(u + v) = f(u - v) = a \neq 0$. Hence the left-hand side of equation (17.1) is not equal to the right-hand side if either $u + \varepsilon v \in \{(0,0), (0,2)\}$ and $u - \varepsilon v \in \{(2,0), (2,2)\}$ or $u - \varepsilon v \in \{(0,0), (0,2)\}$ and $u + \varepsilon v \in \{(2,0), (2,2)\}$. In each of these cases we have $2u \in \{(2,0), (2,2)\}$, and hence $u \in \{(1,0) + Y_{(2)}, (1,1) + Y_{(2)}\}$, contrary to the assumption. Thus both sides of equation (17.1) are equal.

4. $u \pm v \notin Y_{(2)}$, $u \pm \varepsilon v \notin Y_{(2)}$. Since $u + v = u - v + 2v$, the elements $u + v$ and $u - v$ belong simultaneously either to the coset $(1,0) + Y_{(2)}$ or to the coset $(0,1) + Y_{(2)}$ or to the coset $(1,1) + Y_{(2)}$. Hence $f(u + v) = f(u - v)$ and $g(u + \varepsilon v) = g(u - \varepsilon v)$. Then equation (17.1) is satisfied.

We are left to verify that the functions $f(y)$ and $g(y)$ can not be represented as a product of the characteristic functions of distributions of the classes (I)–(III) defined in 17.10. We will prove this for the function $f(y)$. Assume that $f(y) = f_1(y) f_2(y) f_3(y)$, where $f_j(y)$ is the characteristic function of the corresponding distribution. There is no factor $f_3(y)$ in this product, because there exist elements $y \notin Y_{(2)}$ such that $f(y) \neq 0$. Thus $f(y) = f_1(y) f_2(y)$. Suppose that both factors are present in this decomposition. If $y \in Y^{(2)}$, then $f_2(y) = 1$. But the module of $f_1(y)$ equals either 0 or 1. Hence for every $y \in Y^{(2)}$ the module of $f(y)$ equals either 0 or 1. But this is false. Hence either $f(y) = f_1(y)$ or $f(y) = f_2(y)$. It is easy to see that neither case is possible. The arguments for the function $g(y)$ are similar. In view of Lemma 16.1 we have proved the lemma for the group $X = (\mathbb{Z}(4))^2$.

Let $X = (\mathbb{Z}(4))^3$. Denote by (m, n, p), $m, n, p \in \mathbb{Z}(4)$, elements of the group X. Let an automorphism $\delta \in \mathrm{Aut}(X)$ be of the form $\delta(m, n, p) = (m+n+p, m+n, m)$. Obviously, $I \pm \delta \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and the distributions

$$\mu_1 = am_{K_1} + (1-a)m_F, \quad \mu_2 = am_{K_2} + (1-a)m_F,$$

where $K_1 = \{(0,0,0), (0,2,2)\}$, $K_2 = \{(0,0,0), (2,0,0)\}$, $F = \{Y_{(2)}, (0,0,1) + Y_{(2)}\}$, and $0 < a < 1$. Further arguments follow the scheme of the proof of the lemma for the group $X = (\mathbb{Z}(4))^2$ but require consideration of a greater number of cases and we omit them. □

Theorem 17.12. *Let X be a finite 2-prime group such that:*

(i) $X \neq X_{(2)}$;

(ii) *the decomposition of the group X into a direct product of cyclic subgroups contains each cyclic factor with multiplicity not less than* 2.

Then there exist an automorphism $\bar\delta \in \mathrm{Aut}(X)$ such that $I \pm \bar\delta \in \mathrm{Aut}(X)$ and independent random variables ξ_1 and ξ_2 with values in X and distributions μ_1 and μ_2 such that the conditional distribution of the linear form $L_2 = \xi_1 + \bar\delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, whereas no μ_j is represented as a convolution of distributions of the classes (I)–(III) *defined in* 17.10.

Proof. Since $X \neq X_{(2)}$ and condition (ii) is satisfied, the group X can be represented in the form $X = G \times K$, where $G \cong (\mathbb{Z}(2^l))^r$, $l \geq 2$, and either $r = 2$ or $r = 3$. Moreover, the subgroup K also satisfies condition (ii). Assume for definiteness that $r = 2$. Denote by (m, n), $m, n \in \mathbb{Z}(2^l)$, elements of G. Let $\delta \in \mathrm{Aut}(G)$ be of the form $\delta(m, n) = (m + n, m)$. Then $I \pm \delta \in \mathrm{Aut}(G)$. Let F be a subgroup of G such that $F \cong (\mathbb{Z}(4))^2$. Obviously, the restriction of δ to F is an automorphism of F. As has been proved in Lemma 17.11 for a given $\delta \in \mathrm{Aut}(X)$ there exist independent random variables ξ_1 and ξ_2 with values in the subgroup F and distributions μ_1 and μ_2 such that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, whereas no μ_j is represented as a convolution of distributions of the classes (I)–(III) defined in 17.10.

Extend the automorphism δ to an automorphism $\bar\delta \in \mathrm{Aut}(X)$ such that $I \pm \bar\delta \in \mathrm{Aut}(X)$. This can be done because the subgroup K also satisfies condition (ii). We can suppose that the random variables ξ_1 and ξ_2 take values in X. Obviously, the conditional distribution of the linear form $L_2 = \xi_1 + \bar\delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric whereas no μ_j is represented as a convolution of distributions of the classes (I)–(III) defined in 17.10.

In the case when $G \cong (\mathbb{Z}(2^l))^3$ we consider $\delta \in \mathrm{Aut}(G)$ of the form $\delta(m, n, p) = (m + n + p, m + n, m)$, $m, n, p \in \mathbb{Z}(2^l)$ and argue similarly. □

Let $X = \mathbb{R} \times N$, where N is a finite group. Let α_j, β_j be topological automorphisms of X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. We will discuss the following problem.

What are possible distributions μ_j of independent random variables ξ_j taking values in the group X assuming that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric? To solve this problem we need the following

Lemma 17.13. *Let $X = \mathbb{R} \times G$, where G is a finite 2-primary group. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. If the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \gamma_j * \rho_j$, where $\gamma_j \in \Gamma(\mathbb{R})$, $\sigma(\rho_j) \subset G$, $j = 1, 2$.*

Proof. We have $Y \cong \mathbb{R} \times H$, where $H = G^*$. To avoid introducing new notation we will suppose that $Y = \mathbb{R} \times H$. Denote by $y = (s, h)$, $s \in \mathbb{R}$, $h \in H$, elements of the group Y. Since $c_Y = \mathbb{R}$ and $b_Y = H$, the subgroups $\mathbb{R}$ and H of Y are characteristic. Therefore any automorphism $\alpha \in \mathrm{Aut}(Y)$ can be written in the form $\alpha(s, h) = (\alpha s, \alpha h)$, $(s, h) \in Y$. Arguing as in the proof of Theorem 17.1 we reduce the proof of the lemma to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$. Set $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$, $\varepsilon = \tilde{\delta}$. By Lemma 17.1 it follows from the symmetry of the conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ that the characteristic functions $f(y)$ and $g(y)$ satisfy equation (17.1) which takes the form

$$f(s + s', h + h')g(s + \varepsilon s', h + \varepsilon h') \\ = f(s - s', h - h')g(s - \varepsilon s', h - \varepsilon h'), \quad (s, h), (s', h') \in Y. \tag{17.27}$$

Setting $h = h' = 0$ into (17.27), we get the equation

$$f(s + s', 0)g(s + \varepsilon s', 0) = f(s - s', 0)g(s - \varepsilon s', 0), \quad s, s' \in \mathbb{R}.$$

By the Heyde theorem we conclude from this that

$$f(s, 0) = \exp\{-\sigma_1 s^2 + it_1 s\}, \quad g(s, 0) = \exp\{-\sigma_2 s^2 + it_2 s\}, \quad s \in \mathbb{R}, \tag{17.28}$$

where $\sigma_j \geq 0$, $t_j \in \mathbb{R}$, $j = 1, 2$.

We will prove by induction on k, where 2^k is the order of an element h, that

$$f(s, h) = F_1(s)F_2(h), \quad g(s, h) = G_1(s)G_2(h), \tag{17.29}$$

where $F_j(0) = G_j(0) = 1$, $j = 1, 2$.

Substituting $s = -\varepsilon s'$, $h' = h$ into (17.27), we get the equation

$$f(\beta s', 2h)g(0, \alpha h) = f(-\alpha s', 0)g(-2\varepsilon s', \beta h), \quad (s, h), (s', h') \in Y, \tag{17.30}$$

where $\alpha = I + \varepsilon$, $\beta = I - \varepsilon$. Let $k = 1$, i.e., $2h = 0$. Then equation (17.30) takes the form

$$f(\beta s', 0)g(0, \alpha h) = f(-\alpha s', 0)g(-2\varepsilon s', \beta h), \quad (s, h), (s', h') \in Y. \tag{17.31}$$

We deduce from (17.28) that $f(-\alpha s', 0) \neq 0$. Since $-2\varepsilon \in \mathrm{Aut}(\mathbb{R})$ and $\beta \in \mathrm{Aut}(H)$, (17.31) yields the required representation for the function $g(s,h)$.

Substituting $s' = -s$, $h = \varepsilon h'$ into (17.27), we obtain the equation

$$f(0, \alpha h) g(\beta s, 2\varepsilon h') = f(2s, -\beta h') g(\alpha s, 0), \quad (s,h), (s',h') \in Y. \tag{17.32}$$

If $k = 1$, i.e., when $2h' = 0$, the required representation for the function $f(s,h)$ follows from (17.32). Thus for $k = 1$, (17.29) is proved.

Assume that (17.29) holds if h has order 2^k. Let h have order 2^{k+1}. Then $2h$ has order 2^k, and we have $f(\beta s', 2h) = F_1(\beta s') F_2(2h)$ by induction hypothesis. The required representation for the function $g(s,h)$ follows from (17.30). Arguing similarly we obtain the required representation for the function $f(s,h)$ from (17.32).

We conclude from (17.29) that

$$f(s,h) = f(s,0) f(0,h), \;\; g(s,h) = g(s,0) g(0,h). \tag{17.33}$$

Note that $f(0,h)$ and $g(0,h)$ are the characteristic functions of some distributions ρ_1 and ρ_2 such that $\sigma(\rho_j) \in G$, $j = 1, 2$. Taking into account 2.7 (b) and 2.7 (c), the statement of the lemma follows from (17.28) and (17.33). □

Theorem 17.14. *Let $X = \mathbb{R} \times N$, where N is a finite group. Denote by N_2 the 2-component of N. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. If the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_j = \gamma_j * \rho_j * \tau_j$, where $\gamma_j \in \Gamma(\mathbb{R})$, $\sigma(\rho_j) \subset N_2$, $\tau_j \in I(X)$, $j = 1, 2$.*

Proof. Decompose the group N into a direct product of its p-components:

$$N = \mathop{\mathbf{P}}_{p \in \mathcal{P}} N_p.$$

Put

$$G = N_2, \quad K = \mathop{\mathbf{P}}_{p > 2} N_p.$$

Then $X = \mathbb{R} \times G \times K$, and $Y \cong \mathbb{R} \times H \times L$, where $H = G^*$, $L = K^*$. To avoid introducing new notation we will assume that $Y = \mathbb{R} \times H \times L$. Denote by $y = (s,h,l), s \in \mathbb{R}$, $h \in H$, $l \in L$, elements of the group Y. Since $\mathbb{R}$, H, and L are characteristic subgroups of the group Y, any automorphism $\alpha \in \mathrm{Aut}(Y)$ can be written in the form $\alpha(s,h,l) = (\alpha s, \alpha h, \alpha l)$, $(s,h,l) \in Y$. Arguing as in the proof of Theorem 17.1 we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$. Set $f(y) = \hat\mu_1(y)$, $g(y) = \hat\mu_2(y)$, $\varepsilon = \tilde\delta$. By Lemma 17.1 if the conditional distribution of $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the characteristic functions $f(y)$ and $g(y)$ satisfy equation (17.1) which takes the form

$$\begin{aligned} &f(s+s', h+h', l+l') g(s+\varepsilon s', h+\varepsilon h', l+\varepsilon l') \\ &\quad = f(s-s', h-h', l-l') g(s-\varepsilon s', h-\varepsilon h', l-\varepsilon l'), \quad (s,h,l), (s',h',l') \in Y. \end{aligned} \tag{17.34}$$

Substituting $s = s' = 0$ into (17.34), we get

$$f(0, h + h', l + l')g(0, h + \varepsilon h', l + \varepsilon l') = f(0, h - h', l - l')g(0, h - \varepsilon h', l - \varepsilon l').$$

By Corollary 17.9 we can assume from the beginning that the following representations are valid:

$$f(0, h, l) = \begin{cases} f_0(h) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0, \end{cases} \quad g(0, h, l) = \begin{cases} g_0(h) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0, \end{cases} \tag{17.35}$$

where $f_0(h)$ and $g_0(h)$ are some characteristic functions on the group H. Put $\alpha = I + \varepsilon$, $\beta = I - \varepsilon$. Substituting $s' = s$, $h' = -h$, $l' = -l$ into (17.34), we get

$$f(2s, 0, 0)g(\alpha s, \beta h, \beta l) = f(0, 2h, 2l)g(\beta s, \alpha h, \alpha l), \quad (s, h, l) \in Y.$$

We conclude from (17.35) that $f(0, 2h, 2l) = 0$ for $l \neq 0$. Hence

$$f(2s, 0, 0)g(\alpha s, \beta h, \beta l) = 0, \quad l \neq 0. \tag{17.36}$$

Since by the Heyde theorem representation (17.28) holds true for the function $f(s, 0, 0)$, we have $f(2s, 0, 0) \neq 0$. Hence (17.36) implies that

$$g(\alpha s, \beta h, \beta l) = 0, \quad l \neq 0. \tag{17.37}$$

Taking into account that $\alpha, \beta \in \mathrm{Aut}(Y)$, it follows from (17.37) that

$$g(s, h, l) = 0, \quad l \neq 0. \tag{17.38}$$

Arguing similarly, we get

$$f(s, h, l) = 0, \quad l \neq 0. \tag{17.39}$$

Put $l = l' = 0$ in (17.34). Then by Lemma 17.13 we have the representations

$$f(s, h, 0) = \exp\{-\sigma_1 s^2 + i t_1 s\}F(h), \ g(s, h, 0) = \exp\{-\sigma_2 s^2 + i t_2 s\}G(h), \tag{17.40}$$

where $\sigma_j \geq 0$, $t_j \in \mathbb{R}$, $j = 1, 2$.

We deduce from (17.38)–(17.40) that

$$f(s, h, l) = \begin{cases} \exp\{-\sigma_1 s^2 + i t_1 s\}F(h) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0, \end{cases} \tag{17.41}$$

$$g(s, h, l) = \begin{cases} \exp\{-\sigma_2 s^2 + i t_2 s\}G(h) & \text{if } l = 0, \\ 0 & \text{if } \neq 0. \end{cases} \tag{17.42}$$

It is easily seen that the assertion of the theorem follows from 2.7 (b), 2.7 (c), (17.41), and (17.42). □

Now we will discuss the case of an arbitrary number n of independent random variables taking values in a finite group X. We need two lemmas.

Lemma 17.15. *Let X be a finite 5-primary group satisfying the following condition:*

(i) *The decomposition of the group X into a direct product of its cyclic subgroups contains at least one cyclic factor with multiplicity* 1.

Then there are no automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ such that

(ii) $I \pm \alpha, I \pm \beta, \alpha \pm \beta \in \mathrm{Aut}(X)$.

Proof. By Theorem 1.19.4 we have

$$X \cong \mathop{\mathbf{P}}_j (\mathbb{Z}(5^{k_j}))^{n_j}, \quad k_j < k_{j+1}.$$

The numbers k_j and n_j are uniquely determined by the group X. To avoid introducing new notation we will suppose that

$$X = \mathop{\mathbf{P}}_j (\mathbb{Z}(5^{k_j}))^{n_j}, \quad k_j < k_{j+1}.$$

By the condition of the lemma, $n_{j_0} = 1$ for some j_0. Put $G_1 = X^{(5^{k_{j_0}-1})} \cap X_{(5)}$, $G_2 = X^{(5^{k_{j_0}})} \cap X_{(5)}$. Then as is easily seen $G = G_1/G_2 \cong \mathbb{Z}(5)$. Since $X^{(5^l)}$ and $X_{(5)}$ are characteristic subgroups, the subgroups G_1 and G_2 are also characteristic. Therefore any automorphism $\delta \in \mathrm{Aut}(X)$ induces an automorphism $\hat{\delta}$ on the factor group G. Note that each automorphism $\hat{\delta} \in \mathrm{Aut}(G)$ has the form $\hat{\delta} g = kg$, $g \in G$, where $k = 1, 2, 3, 4$. It is clear that there are no automorphisms $\widehat{\alpha}, \hat{\beta} \in \mathrm{Aut}(G)$ satisfying condition (ii). Hence there are no automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ satisfying condition (ii). □

Lemma 17.16. *For each of the groups $X = (\mathbb{Z}(p^r))^2$, $X = (\mathbb{Z}(p^r))^3$, where $p = 3$, $p = 5$, $X = \mathbb{Z}(p^r)$, where p is a prime number, $p \geq 7$, $r \in \mathbb{N}$, there exist automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ satisfying condition 17.15* (ii) *and independent identically distributed random variables ξ_j, $j = 1, 2, 3$, with values in X and distribution $\mu \notin I(X)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric.*

Proof. Note that $Y \cong X$. Let $X = (\mathbb{Z}(p^r))^2$, where either $p = 3$ or $p = 5$. Denote by (k, l), $k, l \in \mathbb{Z}(p^r)$, elements of the groups X and Y. Let automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ be of the form

$$\alpha(k, l) = (k + 2l, 2k + 2l), \quad \beta(k, l) = (2k + 2l, 2k + l), \quad k, l \in \mathbb{Z}(p^r).$$

Obviously, condition 17.15 (ii) is satisfied. Note that

$$\tilde{\alpha}(k, l) = (k + 2l, 2k + 2l), \quad \tilde{\beta}(k, l) = (2k + 2l, 2k + l), \quad k, l \in \mathbb{Z}(p^r). \quad (17.43)$$

Put $\tilde{y} = (1, 0) \in Y$. Let μ be the distribution on the group X such as in Lemma 13.20. Then the characteristic function $f(y) = \hat{\mu}(y)$ is defined by (13.30). It is obvious that

$\mu \notin I(X)$. Let ξ_j, $j = 1, 2, 3$, be independent identically distributed random variables with values in X and distribution μ.

By Lemma 16.1 the conditional distribution of $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric if and only if the characteristic functions of the random variables ξ_j satisfy equation 16.1 (i) which takes the form

$$f(u+v)f(u+\tilde{\alpha}v)f(u+\tilde{\beta}v) = f(u-v)f(u-\tilde{\alpha}v)f(u-\tilde{\beta}v), \quad u, v \in Y. \quad (17.44)$$

Obviously, equation (17.44) holds if $v = 0$. Let $u = (k, l)$, $v = (k', l') \neq 0$. We will verify that the left-hand side of equation (17.44) is equal to zero. Indeed, in view of (17.43) in the opposite case we have

$$\begin{cases} l + l' = 0 & \pmod{p^r}, \\ l + 2k' + 2l' = 0 & \pmod{p^r}, \\ l + 2k' + l' = 0 & \pmod{p^r}. \end{cases}$$

This implies that $k' = l' = 0$, i.e., $v = 0$ contrary to the assumption. Arguing as above we obtain that the right-hand side of equation (17.44) is also equal to zero when $v \neq 0$. Thus the function $f(y)$ satisfies equation (17.44). Hence for the groups $X = (\mathbb{Z}(p^r))^2$, where either $p = 3$ or $p = 5$, the lemma is proved. For the rest of the groups we only indicate automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ and a distribution $\mu \in \mathrm{M}^1(X)$.

Let $X = (\mathbb{Z}(p^r))^3$, where either $p = 3$ or $p = 5$. Denote by (k, l, m), $k, l, m \in \mathbb{Z}(p^r)$ elements of the groups X and Y. Put

$$\alpha(k, l, m) = (k + l + m, k + l, k), \quad \beta(k, l, m) = (2k + l + m, k + 2l, k + m).$$

Set $\tilde{y} = (1, 0, 0) \in Y$ and denote by μ the distribution on the group X such as in Lemma 13.20.

Let $X = \mathbb{Z}(p^r)$, where p is a prime number, $p \geq 7$. Put

$$\alpha x = 2x, \quad \beta x = 4x, \quad x \in X.$$

Denote by $\tilde{y}$ an element of order p^r in Y and by μ the distribution on the group X such as in Lemma 13.20. □

Theorem 17.17. *Let X be a finite group. Represent X as a direct product of its p-components: $X = \mathbf{P}_{p \in \mathcal{P}} X_p$. Assume that X satisfies condition 17.7* (iii). *Then the following statements hold:*

(I) *Let a group X satisfy the conditions:*

  (i) *X contains no elements of order* 2*;*

  (ii) *the decomposition of the group X_5 into a direct product of its cyclic subgroups contains at least one cyclic factor with multiplicity* 1.

Let α_j, β_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be automorphisms of the group X such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables *taking values in X and having distributions μ_j. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all $\mu_j \in I(X)$.*

(II) *If $X_{(2)} \neq \{0\}$, then there exist an automorphism $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$ and independent random variables ξ_1 and ξ_2 with values in X and distributions $\mu_1, \mu_2 \notin I(X)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. If $X_{(2)} = \{0\}$ and condition* (ii) *is not fulfilled, then there exist automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ satisfying condition 17.15* (ii) *and independent random variables ξ_j, $j = 1, 2, 3$, taking values in X and having distributions $\mu_j \notin I(X)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric.*

Proof. (I). By Lemma 17.15 we conclude from condition (ii) that $n = 2$. Since condition (i) is satisfied, the assertion of the theorem follows from Theorem 17.1.

(II). Let $X_{(2)} \neq \{0\}$. Represent X in the form $X = X_2 \times K$, where $K = \mathbf{P}_{p>2}\, X_p$. Since the group X satisfies condition 17.7 (iii), by Proposition 17.7 there exists an automorphism $\delta \in \mathrm{Aut}(X)$ such that $I \pm \delta \in \mathrm{Aut}(X)$. Consider arbitrary independent random variables ξ_1 and ξ_2 with values in $X_{(2)}$ and distributions $\mu_1, \mu_2 \notin I(X_{(2)})$. Taking into account Remark 16.3, we see that the conditional distribution of the linear form $L_2 = \xi_1 + \delta\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.

Let $X_{(2)} = \{0\}$ and condition (ii) be not satisfied. Since the group X satisfies condition 17.7 (iii), obviously, the group X is decomposed in a direct product of groups each of which is isomorphic to one of the groups mentioned in Lemma 17.16. Let G be one of these factors. By Lemma 17.16 there exist automorphisms $\alpha_G, \beta_G \in \mathrm{Aut}(G)$ satisfying condition 17.15 (ii) and independent identically distributed random variables ξ_j, $j = 1, 2, 3$, with values in $G \subset X$ and distribution $\mu \notin I(G)$ such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha_G\xi_2 + \beta_G\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric.

Extend the automorphisms α_G and β_G to automorphisms α and β of the group X in such a way that condition 17.15 (ii) is satisfied for the extended automorphisms. This can be done by Lemma 17.16. We may assume that the random variables ξ_j take values in the group X. Obviously, the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric. □

Remark 17.18. Let X be a finite group satisfying condition 17.7 (iii) and such that $X_{(2)} \neq \{0\}$. If X satisfies condition 17.17 (ii), then by Lemma 17.15, $n = 2$, and it follows from Theorem 17.8 that $\mu_j = \rho_j * \tau_j$, where $\sigma(\rho_j) \subset X_2$, $\tau_j \in I(X)$, $j = 1, 2$.

If the condition 17.17 (ii) for the group X is not satisfied, then there exist automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ satisfying condition 17.15 (ii) and independent random variables ξ_j, $j = 1, 2, 3$, with values in X and distributions μ_j such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2 + \beta\xi_3$ given $L_1 = \xi_1 + \xi_2 + \xi_3$ is symmetric, whereas no μ_j is represented in the form $\mu_j = \rho_j * \tau_j$, where $\sigma(\rho_j) \subset X_2$,

$\tau_j \in I(X)$. Indeed, represent X in the form $X = G \times K$, where G is isomorphic to one of the group mentioned in Lemma 17.16, and the group K also satisfies condition 17.7 (iii) and does not satisfy condition 17.17 (ii). Denote by K_2 the 2-component of the group K. Then we argue as in the proof of Theorem 17.17. The only difference is in the construction of automorphisms $\alpha, \beta \in \mathrm{Aut}(K_2)$ satisfying condition 17.15 (ii).

Since the group X satisfies condition 17.7 (iii), the subgroup K_2 is a direct product of subgroups isomorphic to the groups either of the form $(\mathbb{Z}(2^r))^2$ or $(\mathbb{Z}(2^r))^3$. For a subgroup isomorphic to $(\mathbb{Z}(2^r))^2$ put $\alpha(k,l) = (l, k+l)$, $\beta(k,l) = (k+l, k)$, $k, l \in \mathbb{Z}(2^r)$, and for a subgroup isomorphic to $(\mathbb{Z}(2^r))^3$ put $\alpha(k,l,m) = (k+l+m, k+l, k)$, $\beta(k,l,m) = (l+m, k, k+m)$, $k, l, m \in \mathbb{Z}(2^r)$. It is easily seen that α and β are automorphisms and condition 17.15 (ii) is satisfied.

Now we will study the case when X is a discrete group. First we prove a statement which can be regarded as a group analogue of the Heyde theorem for discrete torsion-free groups.

Theorem 17.19. *Let X be a discrete torsion-free group. Assume that automorphisms α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, of X satisfy the condition $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables with values in X and distributions μ_j. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all μ_j are degenerate distributions.*

Proof. Reasoning as in the proof of Theorem 16.2, we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$, where $\delta_j \in \mathrm{Aut}(X)$. The condition $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$ is transformed into the condition $\delta_i \pm \delta_j \in \mathrm{Aut}(X)$ for all $i \neq j$. By Lemma 16.1 the symmetry of the conditional distribution of $L_2 = \delta_1\xi_1 + \cdots + \delta_n\xi_n$ given $L_1 = \xi_1 + \cdots + \xi_n$ implies that the characteristic functions $\hat\mu_j(y)$ satisfy equation 16.1 (i) which takes the form (16.2). Put $\nu_j = \mu_j * \bar\mu_j$. It follows from 2.7 (c) and 2.7 (d) that $\hat\nu_j(y) = |\hat\mu_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat\nu_j(y)$ also satisfy equation (16.2).

Let U be a neighbourhood of zero of the group Y such that all characteristic functions $\nu_j(y) > 0$ for $y \in U$. Set $\varphi_j(y) = -\ln \hat\nu_j(y)$, $y \in U$. We restrict ourselves to the case when $n = 2$. The case of arbitrary $n \geq 2$ is considered similarly. Let V be a symmetric neighbourhood of zero of the group Y such that for any automorphisms $\lambda_j \in \{I, \tilde\delta_1, \tilde\delta_2\}$ the inclusion

$$\sum_{j=1}^{8} \lambda_j(V) \subset U$$

holds. We conclude from (16.2) that the functions $\varphi_j(y)$ satisfy the equation

$$\varphi_1(u + \tilde\delta_1 v) + \varphi_2(u + \tilde\delta_2 v) - \varphi_1(u - \tilde\delta_1 v) - \varphi_2(u - \tilde\delta_2 v) = 0, \quad u,\ v \in V. \tag{17.45}$$

We use the finite difference method to solve equation (17.45). Let k_1 be an arbitrary element of V. Put $h_1 = \tilde\delta_2 k_1$. Hence $h_1 - \tilde\delta_2 k_1 = 0$. Substitute $u + h_1$ for u and $v + k_1$

for v in equation (17.45). Subtracting equation (17.45) from the resulting equation, we obtain

$$\Delta_{l_{11}}\varphi_1(u+\tilde{\delta}_1 v)+\Delta_{l_{12}}\varphi_2(u+\tilde{\delta}_2 v)-\Delta_{l_{13}}\varphi_1(u-\tilde{\delta}_1 v)=0, \quad u,v\in V, \quad (17.46)$$

where $l_{11}=(\tilde{\delta}_2+\tilde{\delta}_1)k_1$, $l_{12}=2\tilde{\delta}_2 k_1$, $l_{13}=(\tilde{\delta}_2-\tilde{\delta}_1)k_1$. Let k_2 be an arbitrary element of V. Put $h_2=\tilde{\delta}_1 k_2$. Hence $h_2-\tilde{\delta}_1 k_2=0$. Substitute $u+h_2$ for u and $v+k_2$ for v in equation (17.46). Subtracting equation (17.46) from the resulting equation, we get

$$\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1(u+\tilde{\delta}_1 v)+\Delta_{l_{22}}\Delta_{l_{12}}\varphi_2(u+\tilde{\delta}_2 v)=0, \quad u,v\in V, \quad (17.47)$$

where $l_{21}=2\tilde{\delta}_1 k_2$, $l_{22}=(\tilde{\delta}_1+\tilde{\delta}_2)k_2$. Let k_3 be an arbitrary element of V. Put $h_3=-\tilde{\delta}_2 k_3$. Hence $h_3+\tilde{\delta}_2 k_3=0$. Substitute $u+h_3$ for u and $v+k_3$ for v in equation (17.47). Subtracting equation (17.47) from the resulting equation, we obtain

$$\Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1(u+\tilde{\delta}_1 v)=0, \quad u,v\in V, \quad (17.48)$$

where $l_{31}=(\tilde{\delta}_1-\tilde{\delta}_2)k_3$. Putting $v=0$ in (17.48), we find

$$\Delta_{l_{31}}\Delta_{l_{21}}\Delta_{l_{11}}\varphi_1(u)=0, \quad u\in V. \quad (17.49)$$

Since X is a torsion-free group, by Theorems 1.6.1 and 1.6.2, Y is a compact connected group. Then by Theorem 1.9.6, $Y^{(2)}=Y$. Hence the homomorphism $f_2\colon Y\mapsto Y$ is open ([59], (5.9)). We deduce from 1.13 (c) and from the condition of the theorem that $\tilde{\delta}_1\pm\tilde{\delta}_2\in \mathrm{Aut}(Y)$. Taking into account (17.49) and the representations for l_{11}, l_{21}, l_{31}, we conclude that there exists a neighbourhood of zero W of the group Y such that

$$\Delta_h^3\varphi_1(y)=0, \quad h,y\in W. \quad (17.50)$$

Since Y is a compact group, by Theorem 1.12.2 for any neighbourhood of zero W there exists a compact subgroup H of Y such that $H\subset W$ and the factor group $Y/H\cong \mathbb{T}^m\times F$, where $m\geq 0$ and F is a finite group. Since Y is a connected group, $F=\{0\}$, i.e., $Y/H\cong\mathbb{T}^m$. Let $p_1\colon Y\mapsto Y/H$ be the natural homomorphism, and $p_2\colon Y/H\mapsto\mathbb{T}^m$ be the above mentioned isomorphism. Consider the homomorphism $p\colon Y\mapsto\mathbb{T}^m$ defined by $p=p_2\circ p_1$. Since p is an open homomorphism, $p(W)$ is a neighbourhood of zero in $\mathbb{T}^m$. Denote by $t=(t_1,\dots,t_m)$, where $-\pi\leq t_j<\pi$, elements of the group $\mathbb{T}^m$. The operation in $\mathbb{T}^m$ is coordinate-wise addition modulo 2π. Consider the restriction of equation (17.50) to the subgroup H. The function $\varphi_1(y)$ is a continuous polynomial on the group H. Since H is a compact group, by Proposition 5.7, $\varphi_1(y)=\text{const}$ for $y\in H$. Hence $\varphi_1(y)=0$, $y\in H$, and therefore $\hat{\nu}_1(y)=1$, $y\in H$. By Proposition 2.13 it follows from this that $\hat{\nu}_1(y+h)=\hat{\nu}_1(y)$, $y\in Y$, $h\in H$. Thus the function $\hat{\nu}_1(y)$ induces a positive definite function $f_1(t)$ on the group $\mathbb{T}^m$ by the formula $f_1(t)=\hat{\nu}_1(y)$, $t=py$. In view of $\mathbb{T}^m\cong(\mathbb{Z}^m)^*$ by the Bochner theorem there exists a distribution $\gamma_1\in \mathrm{M}^1(\mathbb{Z}^k)$ such that $\hat{\gamma}_1(t)=f_1(t)$,

$t \in \mathbb{T}^m$. Moreover, in the neighbourhood of zero $p(W)$ of the group $\mathbb{T}^k$ we have the representation

$$f_1(t) = e^{-\tilde{\varphi}_1(t)}, \quad t \in p(W), \tag{17.51}$$

where $\tilde{\varphi}_1(t) = \varphi_1(y)$, $t = py$. It follows from (17.50) that $\tilde{\varphi}_1(t)$ is an ordinary polynomial of m variables $t_1, \dots, t_m$. Since $\mathbb{Z}^m \subset \mathbb{R}^m$, we can consider γ_1 as a distribution on $\mathbb{R}^m$ supported in $\mathbb{Z}^m$, i.e., assume that the function $f_1(t)$ is defined on $\mathbb{R}^m$ and $f_1(t)$ is a 2π-periodic function in each variable. It follows from (17.51) that we can apply Theorem 2.19, assuming that $F(t) = f_1(t)$ and $\Phi(t) = e^{-\tilde{\varphi}_1(t)}$. By Theorem 2.19 the left-hand side of (17.51) is an entire function, and representation (17.51) is valid for any $t \in \mathbb{R}^m$. Since $\tilde{\varphi}_1(t)$ is a continuous polynomial and $\tilde{\varphi}_1(t)$ is a 2π-periodic function in each variable, we obtain that $\tilde{\varphi}_1(t) = \text{const}$, $t \in \mathbb{R}^m$. Hence $\tilde{\varphi}_1(t) = 0$, $t \in \mathbb{R}^m$. This implies that $f_1(t) = 1$, $t \in \mathbb{R}^m$, and therefore $\hat{\nu}_1(y) = 1$, $y \in Y$. Thus $\nu_1 = E_0$. Hence μ_1 is a degenerate distribution. Using similar reasoning we show that μ_2 is also a degenerate distribution.

In the case of an arbitrary n we follow the scheme of the proof of the theorem for $n = 2$. □

Corollary 17.20. *Let Y be a compact connected group and $\tilde{\alpha}_j$, $\tilde{\beta}_j$, $j = 1, 2, \dots, n$, $n \geq 2$, be topological automorphisms of Y such that $\tilde{\beta}_i\tilde{\alpha}_i^{-1} \pm \tilde{\beta}_j\tilde{\alpha}_j^{-1} \in \mathrm{Aut}(Y)$ for all $i \neq j$. Let $\hat{\mu}_j(y)$ be characteristic functions on the group Y satisfying equation 16.1* (i). *Then $\hat{\mu}_j(y)$ are of the form*

$$\hat{\mu}_j(y) = (x_j, y), \quad x_j \in X, \ j = 1, 2, \dots, n.$$

Remark 17.21. Arguing as in the proof of Theorem 17.19 it is not difficult to prove that the following statement holds: Let $X = \mathbb{R}^m$, where $m > 1$. Let α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, be topological automorphisms of X such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables with values in X and distributions μ_j. Assume that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric. Then all μ_j are Gaussian distributions. We may consider this assertion as an analogue of the Heyde theorem for the group $X = \mathbb{R}^m$. Taking into account this statement we can prove Lemma 17.13 for the group $X = \mathbb{R}^m \times G$, where $m > 1$ and G is a finite 2-primary group. Applying this generalization of Lemma 17.13 we can prove Theorem 17.14 for the group $X = \mathbb{R}^m \times N$, where $m > 1$ and N is a finite group.

Proposition 17.22. *Let $X = \mathbb{R}^m \times G$, where $m \geq 0$ and the group G contains a compact open subgroup. Let α_j, β_j, $j = 1, 2, \dots, n$, $n \geq 2$, be topological automorphisms of X such that $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \neq j$. Let ξ_j be independent random variables with values in the group X and distributions μ_j. Assume that the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric. Then the random variables ξ_j can be replaced by their shifts ξ_j' in such a way that all distributions μ_j' are supported in $\mathbb{R}^m \times b_G$, and the conditional distribution of the linear form $L_2' = \beta_1\xi_1' + \cdots + \beta_n\xi_n'$ given $L_1' = \alpha_1\xi_1' + \cdots + \alpha_n\xi_n'$ is symmetric.*

Proof. Let c_Y be the connected component of zero of the group Y. By Theorem 1.11.2, $c_Y = M \times L$, where $M \cong \mathbb{R}^m$ and L is a compact connected group. By Lemma 16.1 the symmetry of the conditional distribution of the linear form L_2 given L_1 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i). Obviously, L is a characteristic subgroup of the group Y. Therefore we can consider the restriction of equation 16.1 (i) to the subgroup L. Applying Corollary 17.20 to L and using Theorem 1.9.2, we have the representation

$$\hat{\mu}_j(y) = (x_j, y), \quad y \in L, \ j = 1, 2, \dots, n. \tag{17.52}$$

Substituting (17.52) into 16.1 (i), we conclude that

$$2\sum_{j=1}^{n} \beta_j x_j \in A(X, L). \tag{17.53}$$

Note that $A(X, L) = \mathbb{R}^m \times b_G$. By Theorem 1.9.6 we have $L^{(2)} = L$. It follows from (17.46) and Proposition 7.4 that $x_0 = \sum_{j=1}^{n} \beta_j x_j \in \mathbb{R}^m \times b_G$. Obviously, the subgroup $\mathbb{R}^m \times b_G$ is characteristic. Put $x'_1 = x_1 - \beta_1^{-1} x_0$, $x'_j = x_j$, $j = 2, \dots, n$. Let μ'_j be a distribution on the group X with the characteristic function $\hat{\mu}'_j(y) = (-x'_j, y)\hat{\mu}_j(y)$, $j = 1, 2, \dots, n$. Since $\beta_1^{-1} x_0 \in \mathbb{R}^m \times b_G = A(X, L)$, we have

$$\hat{\mu}_j(y) = (x'_j, y), \quad y \in L, \ j = 1, 2, \dots, n.$$

Moreover, $\sum_{j=1}^{n} \beta_j x'_j = 0$. This implies that the characteristic functions $f_j(y) = (-x'_j, y)$ satisfy equation 16.1 (i). Hence the characteristic functions $\hat{\mu}'_j(y)$ also satisfy equation 16.1 (i). Let ξ'_j be independent random variables with values in the group X and distributions μ'_j. By Lemma 16.1 the conditional distribution of the linear form $L'_2 = \beta_1 \xi'_1 + \cdots + \beta_n \xi'_n$ given $L'_1 = \alpha_1 \xi'_1 + \cdots + \alpha_n \xi'_n$ is symmetric. Since

$$\hat{\mu}'_j(y) = 1, \quad y \in L, \ j = 1, 2, \dots, n,$$

we have by Proposition 2.13, $\sigma(\mu'_j) \subset A(X, L) = \mathbb{R}^m \times b_G$. □

Remark 17.23. Let $X = \mathbb{R}^m \times G$ where $m \ge 0$ and the group G contains a compact open subgroup. Since $\mathbb{R}^m \times b_G$ is a characteristic subgroup, Proposition 17.22 implies the following statement: Let ξ_j, $j = 1, 2, \dots, n$, $n \ge 2$, be independent random variables with values in X and distributions μ_j. Let α_j, β_j be topological automorphisms of the group X such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \mathrm{Aut}(X)$ for all $i \ne j$. Assume that the conditional distribution of the linear form $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ is symmetric. Then in studying the possible distributions μ_j we can assume without loss of generality that $G = b_G$, i.e., the group G itself consists of compact elements.

Lemma 17.24. *Let X be a discrete torsion group containing no elements of order 2. Let α_j, β_j, $j = 1, 2, \dots, n$, $n \ge 2$, be topological automorphisms of X such that*

$\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \in \mathrm{Aut}(X)$ *for all* $i \neq j$. *Let* ξ_j *be independent random variables with values in the group* X *and distributions* μ_j *such that all characteristic functions* $\hat{\mu}_j(y) \geq 0$. *If the conditional distribution of the linear form* $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ *given* $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ *is symmetric, then all characteristic functions* $\hat{\mu}_j(y) \equiv 1$ *on an open subgroup* $B \subset Y$.

Proof. Since X is a discrete torsion group, by Theorems 1.6.1 and 1.6.4, Y is compact and totally disconnected. The compactness of the group Y implies that $\overline{Y^{(2)}} = Y^{(2)}$. Since $X_{(2)} = \{0\}$, by Theorem 1.9.2, $\overline{Y^{(2)}} = Y$. It follows from this that $Y^{(2)} = Y$, and hence the homomorphism $f_2 \colon Y \mapsto Y$ is open ([59], (5.9)). We restrict ourselves to the case $n = 2$. Arguing as in the proof of Theorem 17.19 we come to equation (17.50) for the function $\varphi_1(y) = -\ln \hat{\mu}_1(y)$ in a neighbourhood W of zero of the group Y. By Theorem 1.12.1, W contains a compact open subgroup B_1. Since B_1 is open, it is closed, and hence B_1 is compact. Therefore, by Proposition 5.7 we have $\varphi_1(y) = 0$ for $y \in B_1$. This implies that $\hat{\mu}_1(y) = 1$ for $y \in B_1$. Arguing similarly for the distribution μ_2 we find a subgroup B_2 such that $\hat{\mu}_2(y) = 1$ for $y \in B_2$. Put $B = B_1 \cap B_2$. □

Lemma 17.25. *Let* ξ_1 *and* ξ_2 *be independent random variables with values in a group* X *and distributions* $\mu_1 = m_{K_1}$ *and* $\mu_2 = m_{K_2}$, *where* K_1 *and* K_2 *are finite subgroups of* X. *Let* $f_2, \delta, I \pm \delta \in \mathrm{Aut}(X)$. *If the conditional distribution of the linear form* $L_2 = \xi_1 + \delta\xi_2$ *given* $L_1 = \xi_1 + \xi_2$ *is symmetric, then* $K_1 = K_2 = K$ *and* $\delta(K) = K$.

Proof. Set $f(y) = \widehat{m}_{K_1}(y), g(y) = \widehat{m}_{K_2}(y), \varepsilon = \tilde{\delta}, \alpha = I + \varepsilon, \beta = I - \varepsilon, \kappa = \beta\alpha^{-1}$. Then $\kappa = \tilde{\gamma}$, where $\gamma = (I+\delta)^{-1}(I-\delta)$. By Lemma 16.1 it follows from the symmetry of the conditional distribution of L_2 given L_1 that the characteristic functions $f(y)$ and $g(y)$ satisfy equation 16.1 (i) which takes the form (17.1). Equation (17.1) yields (17.2). Put $H_j = A(Y, K_j)$, $j = 1, 2$. We conclude from (17.2) that if $\kappa y \in H_2$, then $y \in H_2$. By Lemma 13.10 this implies that $\gamma(K_2) \supset K_2$. Since K_2 is a finite group, we have

$$\gamma(K_2) = K_2. \tag{17.54}$$

Note that $I + \gamma = f_2(I + \delta)^{-1}$, $I - \gamma = f_2\delta(I + \delta)^{-1}$. Since $f_2 \in \mathrm{Aut}(X)$, we have $I \pm \gamma \in \mathrm{Aut}(X)$ and $\delta = (I - \gamma)(I + \gamma)^{-1}$. We deduce from (17.54) that $\delta(K_2) = K_2$, and by Lemma 13.11, $\varepsilon(H_2) = H_2$. Consider the restriction of equation (17.1) to the subgroup H_2. We get

$$f(u + v) = f(u - v), \quad u, v \in H_2.$$

Hence

$$f(2y) = 1, \quad y \in H_2. \tag{17.55}$$

Since $f_2 \in \mathrm{Aut}(X)$ and K_2 is a finite group, we have $(K_2)^{(2)} = K_2$, and hence by Lemma 13.11, $(H_2)^{(2)} = H_2$. Now (17.55) implies that $f(y) = 1$ for $y \in H_2$. Therefore, $H_2 \subset H_1$. Arguing as above we get from (17.1) that $\varepsilon(H_1) = H_1$ and

$$g(2\varepsilon y) = 1, \quad y \in H_1,$$

holds. This implies that $H_1 \subset H_2$. Thus $H_1 = H_2 = H$, and hence $K_1 = K_2 = K$. Since $\varepsilon(H) = H$, by Lemma 13.11, $\delta(K) = K$. □

Theorem 17.26. *Let X be a discrete group containing no elements of order 2. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in$ Aut(X). If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_1, \mu_2 \in I(X)$.*

Proof. Arguing as in the proof of Theorem 17.1, we reduce the proof of the theorem to the case when $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + \delta\xi_2$, where $\delta, I \pm \delta \in \mathrm{Aut}(X)$. Set $\varepsilon = \tilde{\delta}$. By Lemma 16.1 the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation 16.1 (i). Put $f(y) = \hat{\mu}_1(y)$, $g(y) = \hat{\mu}_2(y)$ and rewrite equation 16.1 (i) using this notation. We get equation (17.1). Put $\nu_j = \mu_j * \bar{\mu}_j$. We conclude from 2.7 (c) and 2.7 (d) that $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0$, $y \in Y$. Obviously, the characteristic functions $\hat{\nu}_j(y)$ also satisfy equation (17.1). If we prove that $\nu_j \in I(X)$, then by 2.7 (b) and 2.7 (e) $\mu_j \in I(X)$. Thus we will solve equation (17.1) assuming that $f(y) \geq 0$, $g(y) \geq 0$, $f(-y) = f(y)$, $g(-y) = g(y)$. We will prove that in this case $f(y) = g(y) = \widehat{m}_K(y)$, where K is a subgroup of X. The statement of the theorem follows from this.

Taking into account Remark 17.21, we can assume from the beginning that X is a torsion group. Put $E_f = \{y \in Y : f(y) = 1\}$, $E_g = \{y \in Y : g(y) = 1\}$. Then by Proposition 2.13, $\sigma(\mu_1) \subset A(X, E_f) = F$, $\sigma(\mu_2) \subset A(X, E_g) = G$. It follows from Lemma 17.24 that there exists an open subgroup B such that $B \subset E_f \cap E_g$. Set $S = A(X, B)$. Then F and G are subgroups of S. Since the subgroup L is open, by Theorem 1.9.4, S is a compact group. Taking into account that the group X is discrete, we deduce that S is a finite group. Hence F and G are also finite groups.

Note now that for all natural n the characteristic functions $f^n(y)$ and $g^n(y)$ also satisfy equation (17.1), i.e.,

$$f^n(u + v)g^n(u + \epsilon v) = f^n(u - v)g^n(u - \epsilon v), \quad u, v \in Y. \tag{17.56}$$

Obviously, there exist the limits

$$\bar{f}(y) = \lim_{n\to\infty} f^n(y) = \begin{cases} 1 & \text{if } y \in E_f, \\ 0 & \text{if } y \notin E_f, \end{cases} \quad \bar{g}(y) = \lim_{n\to\infty} g^n(y) = \begin{cases} 1 & \text{if } y \in E_g, \\ 0 & \text{if } y \notin E_g. \end{cases}$$

Note that by Theorem 1.9.1, $E_f = A(Y, F)$, $E_g = A(Y, G)$. Then it follows from 2.14 (i) that

$$\widehat{m}_F(y) = \begin{cases} 1 & \text{if } y \in E_f, \\ 0 & \text{if } y \notin E_f, \end{cases} \quad \widehat{m}_G(y) = \begin{cases} 1 & \text{if } y \in E_g, \\ 0 & \text{if } y \notin E_g. \end{cases}$$

Hence

$$\widehat{m}_F(y) = \bar{f}(y), \quad \widehat{m}_G(y) = \bar{g}(y).$$

Let ζ_1 and ζ_2 be independent random variables with values in the group X and distributions $\lambda_1 = m_F$ and $\lambda_2 = m_G$. We conclude from (17.56) that the characteristic functions $\bar{f}(y)$ and $\bar{g}(y)$ also satisfy equation (17.1). By Lemma 16.1 this implies that the conditional distribution of the linear forms $L_2 = \zeta_1 + \delta\zeta_2$ given $L_1 = \zeta_1 + \zeta_2$ is symmetric. All conditions of Lemma 17.25 are satisfied. By Lemma 17.25, $F = G$ and $\delta(G) = G$.

Let us return to the original random variables ξ_1 and ξ_2 and to the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$. Since $\sigma(\mu_j) \subset G$, the random variables ξ_j take values in the finite group G. Since $\delta(G) = G$, the conditions of Corollary 17.2 are fulfilled. By Corollary 17.2, $\mu_j = m_K * E_{x_j}$, where K is a subgroup of the group G, and $x_j \in X$, $j = 1, 2$. □

Remark 17.27. Theorem 17.26 and Remark 17.21 imply the following assertion. Let $X = \mathbb{R}^m \times N$, where $m \geq 0$ and N is a discrete group containing no elements of order 2. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. Let α_j, β_j, $j = 1, 2$, be topological automorphisms of the group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. If the conditional distribution of the linear form $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ is symmetric, then $\mu_j = \gamma_j * \pi_j$, where $\gamma_j \in \Gamma(\mathbb{R}^m)$, $\pi_j \in I(X)$, $j = 1, 2$.

Appendix
The Kac–Bernstein and Skitovich–Darmois functional equations on locally compact Abelian groups

Let X be a second countable locally compact Abelian group, Y be its character group. Consider equation 10.1 (i) (the Skitovich–Darmois functional equation). We conclude from Theorem 10.3 and Remark 10.4 that the following statements are equivalent:

(i) for any topological automorphisms $\tilde{\alpha}_j$, $\tilde{\beta}_j$ of the group Y all solutions of the Skitovich–Darmois functional equation in the class of non-vanishing continuous normalized positive definite functions are characteristic functions of Gaussian distributions;

(ii) the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$.

In this appendix we give a complete description of groups Y which possess the property: for $n = 2$ all solutions of the Skitovich–Darmois functional equation in the class of non-vanishing continuous normalized Hermitian functions are functions of the Gaussian type. Then we solve an analogous problem in the class of measurable functions (with respect to the Haar measure) and consider some similar problems.

A.1 Definitions and notation. We will assume that X is a second countable locally compact Abelian group. Denote by Y its character group. An automorphism $\alpha \in \mathrm{Aut}(X)$ is called *regular* if the equation $\alpha x = x$, $x \in X$, has a unique solution $x = 0$. A group X is called *regularly complete* (see [68]) if all its topological automorphisms except the identity automorphism are regular. Denote by bX the *Bohr compactification* of a group X. For a given group Y denote by Y_d this group with the discrete topology. We observe that $(bX)^* \cong Y_d$ ([59], (26.12)). If $x_1, \dots, x_k \in X$, then denote by $\langle x_1, \dots, x_k \rangle$ the subgroup of the group X generated by elements x_j, i.e., the set of elements of the form $x = l_1x_1 + \dots + l_kx_k$, where $l_j \in \mathbb{Z}$. A function $f(y)$ on the group Y is called *Hermitian* if

$$f(-y) = \overline{f(y)}, \quad y \in Y.$$

We conclude from the Bochner theorem and 2.7 (d) that each positive definite function is Hermitian.

First we will discuss the following problem: When are all non-vanishing continuous normalized Hermitian solutions of the Kac–Bernstein functional equation functions of the Gaussian type?

Proposition A.2. *The following statements are equivalent:*

(a) *all non-vanishing continuous normalized Hermitian functions $f_j(y)$ on Y satisfying the equation*

$$\text{(i)}\quad f_1(u+v)f_2(u-v) = f_1(u)f_1(v)f_2(u)f_2(-v), \quad u, v \in Y,$$

are represented in the form

$$\text{(ii)}\quad f_j(y) = (x_j, y)\exp\{\varphi(y)\}, \quad y \in Y,$$

where $x_j \in X$ and $\varphi(y)$ is a continuous real-valued function satisfying equation 2.14 (ii);

(b) *X contains no elements of order* 2.

Proof. (a) $\Longrightarrow$ (b). Assume that a group X contains an element of order 2. Take $x_0 \in X_{(2)}$, $x_0 \neq 0$. Then $G = \langle x_0 \rangle \cong \mathbb{Z}(2)$. Set $H = A(Y, G)$. Since $G \subset X_{(2)}$, we conclude from Theorems 1.9.1 and 1.9.5 that $H \supset Y^{(2)}$. Let $a \in \mathbb{R}$, $a \neq 0$, $a \neq \pm 1$. Consider on the group Y the functions

$$f_1(y) = \begin{cases} 1 & \text{if } y \in H, \\ a & \text{if } y \notin H, \end{cases} \qquad f_2(y) = \begin{cases} 1 & \text{if } y \in H, \\ 1/a & \text{if } y \notin H. \end{cases}$$

Since the subgroup H is the annihilator of a compact subgroup, it follows from Theorem 1.9.4 that the subgroup H is open in Y. Hence $f_j(y)$ are continuous functions. It is obvious that the functions $f_j(y)$ do not vanish, and they are normalized and Hermitian. We will verify that the functions $f_j(y)$ satisfy equation (i). It is clear that the right-hand side of equation (i) is equal to 1 for all $u, v \in Y$. Assume that there exist elements $u, v \in Y$ such that the left-hand side of equation (i) is not equal to 1. Then either $u + v \in H$, $u - v \notin H$ or $u + v \notin H$, $u - v \in H$. In both cases we have $2u \notin H$ contrary to the inclusion $H \supset Y^{(2)}$. Thus the functions $f_j(y)$ satisfy equation (i). Obviously, they can not be represented in the form (ii).

(b) $\Longrightarrow$ (a). Assume that $X_{(2)} = \{0\}$. Then by Theorem 1.9.5,

$$Y = \overline{Y^{(2)}}. \tag{A.1}$$

By Remark 7.15 we have on the subgroup $\overline{Y^{(2)}}$ the representation

$$f_j(y) = (x_j, y)\exp\{\varphi(y)\}, \quad y \in \overline{Y^{(2)}}, \tag{A.2}$$

where $x_j \in X$ and the function $\varphi(y)$ is continuous and satisfies equation 2.14 (ii). The desired representation follows from (A.1) and (A.2). □

A.3. Suppose that in equation A.2 (i), $f_1(y) = f_2(y) = f(y)$, i.e., the function $f(y)$ satisfies the equation

$$f(u+v)f(u-v) = f^2(u)f(v)f(-v), \quad u, v \in Y. \tag{A.3}$$

We will solve now the following problem: For which groups Y are all non-vanishing continuous normalized Hermitian solutions of equation (A.3) of the form

$$f(y) = (x, y)\exp\{\varphi(y)\}, \tag{A.4}$$

where $x \in X$, and $\varphi(y)$ is a continuous real-valued function satisfying equation 2.14 (ii)? It turns out that the corresponding class of groups is wider than the class of groups described in Proposition A.2.

Proposition A.4. *The following statements are equivalent:*

(a) *every non-vanishing continuous normalized Hermitian function $f(y)$ on Y satisfying equation* (A.3) *is represented in the form* (A.4);

(b) *X contains at the most one element of order* 2.

Proof. (a) $\Longrightarrow$ (b). Assume that a group X contains more than one element of order 2. Take $x_1, x_2 \in X_{(2)}$, $x_j \neq 0$, $x_1 \neq x_2$, and consider the subgroup $G = \langle x_1,\ x_2\rangle$, generated by elements x_1 and x_2. Then $G \cong (\mathbb{Z}(2))^2$. Set $H = A(Y, G)$. Since $G \subset X_{(2)}$, Theorems 1.9.1 and 1.9.5 imply $H \supset Y^{(2)}$. Consider on the group Y the function

$$f(y) = \begin{cases} 1 & \text{if } y \in H, \\ -1 & \text{if } y \notin H. \end{cases}$$

Since the subgroup H is the annihilator of a compact subgroup, Theorem 1.9.2 implies that the subgroup H is open in Y. Hence $f(y)$ is a continuous function. It is obvious that the function $f(y)$ does not vanish, and it is normalized and Hermitian. Arguing as in the proof of Proposition A.2, we verify that the function $f(y)$ satisfies equation (A.3). By Theorem 1.9.4, $G^* \cong Y/H$. Hence $Y/H \cong (\mathbb{Z}(2))^2$. It follows from this that the function $f(y)$ is not a character of the group Y and therefore it can not be represented in the form (A.4).

(b) $\Longrightarrow$ (a). Assume that either $X_{(2)} = \{0\}$ or $X_{(2)} \cong \mathbb{Z}(2)$. Set $\varphi(y) = \ln|f(y)|$. Since $f(y)$ is a Hermitian function, the equality $|f(-y)| = |f(y)|$ holds. Taking this into account we conclude from equation (A.3) that $\varphi(y)$ is a continuous real-valued function satisfying equation 2.14 (ii).

Put $l(y) = f(y)/|f(y)|$. It is obvious that $l(y)$ is a continuous function satisfying equation 9.5 (ii). Moreover the conditions: $l(-y) = \overline{l(y)}$, $|l(y)| = 1$, for all $y \in Y$, $l(0) = 1$ are satisfied. Therefore Corollary 9.13 yields that $l(y)$ is a character of the group Y. By the Pontryagin duality theorem the function $l(y)$ can be represented in the form $l(y) = (x, y)$, where $x \in X$. Thus the function $f(y)$ is represented in the form (A.4). □

We will discuss now the problem: When are non-vanishing measurable normalized Hermitian solutions of the Kac–Bernstein functional equation functions of the Gaussian type? We need the following well-known statements.

Lemma A.5. *Let $\varphi(y)$ be a measurable function on a group Y satisfying equation* 2.14 (ii). *Then $\varphi(y)$ is continuous.*

Proof. Obviously, any function $\varphi(y)$ satisfying equation 2.14 (ii) is an algebraic polynomial on Y. Since $\varphi(y)$ is a measurable function and any measurable algebraic polynomial on a locally compact Abelian group is continuous (see e.g. [99], Theorem 3.11), the function $\varphi(y)$ is continuous. □

Lemma A.6 ([59], (22.19)). *Let $l(y)$ be a measurable function on a group Y satisfying the equation*

$$\text{(i)} \quad l(u+v) = l(u)l(v), \quad u, v \in Y.$$

Then $l(y)$ is continuous.

Proposition A.7. *The following statements are equivalent:*

(a) *all non-vanishing measurable normalized Hermitian functions $f_j(y)$ on Y satisfying equation* A.2 (i) *are represented in the form* A.2 (ii);

(b) *bX contains no elements of order* 2.

Proof. (a) ⟹ (b). Since $(bX)^* \cong Y_d$, it follows from Theorem 1.9.5 that the condition $(bX)_{(2)} = \{0\}$ is equivalent to the condition $Y = Y^{(2)}$. Hence if $(bX)_{(2)} \neq \{0\}$, then $Y \neq Y^{(2)}$. Let $a \in \mathbb{R}$, $a \neq 0$, $a \neq \pm 1$. Consider on the group Y the functions

$$f_1(y) = \begin{cases} 1 & \text{if } y \in Y^{(2)}, \\ a & \text{if } y \notin Y^{(2)}, \end{cases} \qquad f_2(y) = \begin{cases} 1 & \text{if } y \in Y^{(2)}, \\ 1/a & \text{if } y \notin Y^{(2)}. \end{cases}$$

Since the group Y is second countable, the subgroup $Y^{(2)}$ is measurable. It is obvious that the functions $f_j(y)$ do not vanish, and they are measurable normalized and Hermitian. Arguing as in the proof of Proposition A.2, we verify that the functions $f_j(y)$ satisfy equation A.2 (i). Obviously, the functions $f_j(y)$ can not be represented in the form A.2 (ii). Note that the subgroup $Y^{(2)}$ does not need to be open. Hence the functions $f_j(y)$ may be discontinuous.

(b) ⟹ (a). Assume that $(bX)_{(2)} = \{0\}$. As has been noted above, it follows from this that $Y = Y^{(2)}$. By Remark 7.15 on the subgroup $Y^{(2)}$ we have the representation

$$f_j(y) = l_j(y) \exp\{\varphi(y)\}, \quad y \in Y^{(2)}, \tag{A.5}$$

where each of the functions $l_j(y)$ satisfies equation A.6 (i) on the subgroup $Y^{(2)}$. Hence the representation (A.5) holds on Y. Since $\varphi(y) = \ln |f_j(y)|$, $j = 1, 2$, the function $\varphi(y)$ is measurable. Hence by Lemma A.5 the function $\varphi(y)$ is continuous. By Lemma A.6 the function $l_j(y)$ is also continuous, i.e., $l_j(y)$ is a character of the group Y. Applying the Pontryagin duality theorem we obtain that the function $l_j(y)$ can be represented in the form $l_j(y) = (x_j, y)$, where $x_j \in X$, $j = 1, 2$. □

Proposition A.8. *The following statements are equivalent:*

(a) *every non-vanishing measurable normalized Hermitian function $f(y)$ on Y satisfying equation* (A.3) *is represented in the form* (A.4);

(b) *bX contains at most one element of order* 2.

Proof. (a) $\Longrightarrow$ (b). Assume that the subgroup $(bX)_{(2)}$ contains more than one element of order 2. Since $(bX)^* \cong Y_d$, it follows from Theorems 1.9.2 and 1.9.5 that $((bX)_{(2)})^* \cong Y_d/(Y_d)^{(2)}$. Hence the decomposition of Y with respect to the subgroup $Y^{(2)}$ contains at least four cosets. Consider on the group Y the function

$$f(y) = \begin{cases} 1 & \text{if } y \in Y^{(2)}, \\ -1 & \text{if } y \notin Y^{(2)}. \end{cases}$$

Since the group Y is second countable, the subgroup $Y^{(2)}$ is measurable. It is obvious that the function $f(y)$ does not vanish, and it is measurable normalized and Hermitian. Arguing as in the proof of Proposition A.2, we verify that the function $f(y)$ satisfies equation (A.3). Since the decomposition of the group Y with respect to the subgroup $Y^{(2)}$ contains at least four cosets, it is easily seen that the function $f(y)$ does not satisfy equation A.6 (i) and, hence it can not be represented in the form (A.4).

(b) $\Longrightarrow$ (a). Assume that either $(bX)_{(2)} = \{0\}$ or $(bX)_{(2)} \cong \mathbb{Z}(2)$. The representation $|f(y)| = \exp\{\varphi(y)\}$, where the function $\varphi(y)$ satisfies equation 2.14 (ii), follows from the fact that the function $f(y)$ is Hermitian and satisfies equation (A.3). Since the function $\varphi(y)$ is measurable, we conclude from Lemma A.5 that the function $\varphi(y)$ is continuous.

Put $l(y) = f(y)/|f(y)|$. Let us now make use of Remark 7.15 for $f_1(y) = f_2(y) = f(y)$. By Remark 7.15 we have the representation (A.5).

Assume first that $(bX)_{(2)} = \{0\}$. Since $(bX)^* \cong Y_d$, we deduce from Theorem 1.9.5 that the condition $(bX)_{(2)} = \{0\}$ is equivalent to the condition $Y = Y^{(2)}$. Hence the function $l(y)$ satisfies equation A.6 (i) on the group Y. By Lemma A.6 the function $l(y)$ is continuous. By the Pontryagin duality theorem the function $l(y)$ can be represented in the form $l(y) = (x, y)$, where $x \in X$. Thus the function $f(y)$ can be represented in the form (A.4). In this case (b) $\Longrightarrow$ (a) is proved.

Assume now that $(bX)_{(2)} \cong \mathbb{Z}(2)$. Since $(bX)^* \cong Y_d$, it follows from Theorems 1.9.2 and 1.9.5 that the condition $(bX)_{(2)} \cong \mathbb{Z}(2)$ is equivalent to the fact that the decomposition of the group Y with respect to the subgroup $Y^{(2)}$ contains two cosets, i.e., $Y = Y^{(2)} \cup (y_0 + Y^{(2)})$. Since Y is second countable, the subgroup $Y^{(2)}$ is measurable. The Haar measure of the subgroup $Y^{(2)}$ can not be equal to zero because in this case the Haar measure of the group Y is also equal to zero. On the other hand, if the Haar measure of the subgroup $Y^{(2)}$ is positive, then $Y^{(2)}$ contains a neighborhood of zero of the group Y ([59], (20.17)). Thus the subgroup $Y^{(2)}$ is open. Hence $Y^{(2)}$ is closed and therefore it is locally compact. The function $l(y)$ is measurable and satisfies equation A.6 (i) on a locally compact Abelian group. By Lemma A.6 the function $l(y)$ is continuous on the subgroup $Y^{(2)}$. Hence it is a character of the subgroup $Y^{(2)}$. By Theorem 1.9.2 there exists an element $x_0 \in X$ such that $l(y) = (x_0, y)$, $y \in Y^{(2)}$. Set $m(y) = l(y)/(x_0, y)$. In order to prove that $m(y)$ is a character of the group Y and hence $l(y)$ is a character of the group Y, we argue in the same way as in the proof of the corresponding statement of Proposition 9.11. □

Remark A.9. As has been shown in Lemmas 7.8 and 7.9, the following classes of groups Y coincide:

(a) the class of groups Y on which all non-vanishing continuous normalized positive definite functions $f_j(y)$ satisfying equation A.2 (i) are of the form A.2 (ii);

(b) the class of groups Y such that the group X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$.

Comparing this statement with Propositions A.2 and A.7 we see that when we strengthen the restrictions for the non-vanishing solutions on equation A.2 (i) (measurable, continuous, continuous positive definite) the corresponding class of groups on which these solutions are of the form A.2 (ii) is enlarged (bX contains no elements of order 2, X contains no elements of order 2, X contains no subgroup topologically isomorphic to $\mathbb{T}$). The same is also true for equation (A.3) (see Propositions A.3 and A.8), because as has been proved in Lemmas 9.7 and 9.9 the following classes of groups Y coincide:

(a′) the class of groups Y on which any non-vanishing continuous normalized positive definite function $f(y)$ satisfying equation (A.3) is of the form (A.4);

(b′) the class of groups Y such that the group X contains no subgroup topologically isomorphic to the group $\mathbb{T}^2$.

A.10. Consider equation 10.1 (i) (the Skitovich–Darmois functional equation) in the case $n = 2$. We will study now when all non-vanishing continuous normalized Hermitian solutions of this equation are functions of the Gaussian type. It is easily seen that for $n = 2$ the study of solutions of equation 10.1 (i) is reduced to the study of solutions of equation (10.23), i.e.,

$$f_1(u + v) f_2(u + \varepsilon v) = f_1(u) f_1(v) f_2(u) f_2(\varepsilon v), \quad u, v \in Y. \tag{A.6}$$

Theorem A.11. *Assume that Y is not topologically isomorphic to the group $\mathbb{Z}(2)$. Let ε be an arbitrary topological automorphism of Y, $\varepsilon \neq I$. The following statements are equivalent:*

(a) *all non-vanishing continuous normalized Hermitian functions $f_j(y)$ on Y satisfying equation (A.6) are represented in the form*

$$\text{(i)} \quad f_j(y) = (x_j, y) \exp\{\varphi_j(y)\}, \quad y \in Y,$$

where $x_j \in X$ and $\varphi_j(y)$ are continuous real-valued functions satisfying equation 2.14 (ii);

(b) *X is regularly complete.*

Proof. (a) $\Longrightarrow$ (b). Assume that X is not regularly complete. We will verify that there exist an automorphism $\varepsilon \in \mathrm{Aut}(Y)$, $\varepsilon \neq I$, and non-vanishing continuous normalized Hermitian functions $f_j(y)$ on the group Y satisfying equation (A.6) and such that they can not be represented in the form (i). By the condition there exists a non-regular automorphism $\delta \in \mathrm{Aut}(X)$, $\delta \neq I$. Set $G = \mathrm{Ker}(\delta - I)$, $H = A(Y, G)$ and $\varepsilon = \tilde{\delta}$. Since $G \neq \{0\}$, it follows from 1.13 (b) that

$$H = \overline{(\varepsilon - I)Y} \neq Y. \tag{A.7}$$

Consider the factor group Y/H. We conclude from (A.7) that $Y/H \neq \{0\}$. The inclusion $\varepsilon(H) \subset H$ implies that ε induces a homomorphism $\hat{\varepsilon}$ on the factor group Y/H. Since $\varepsilon y - y \in H$ for any $y \in Y$, we have $\hat{\varepsilon}[y] = [y]$ for any $[y] \in Y/H$, i.e., $\hat{\varepsilon} = I$. Let $h([y])$ be an arbitrary non-vanishing continuous normalized Hermitian function on the factor group Y/H. Set

$$f_1(y) = h([y]), \quad f_2(y) = 1/h([y]).$$

It is obvious that $f_j(y)$ are non-vanishing continuous normalized Hermitian functions on the group Y. Since $\hat{\varepsilon} = I$, the functions $f_j(y)$ satisfy equation (A.6). It is clear that the function $h([y])$ can be chosen in such a way that the functions $f_j(y)$ are not represented in the form (i).

(b) $\Longrightarrow$ (a). Assume that X is regularly complete. Take an arbitrary automorphism $\varepsilon \in \mathrm{Aut}(Y)$, $\varepsilon \neq I$. Since $\delta = \tilde{\varepsilon}$ is a regular automorphism, we have $G = \mathrm{Ker}(\delta - I) = \{0\}$ and hence by 1.13 (b),

$$A(Y, G) = \overline{(\varepsilon - I)Y} = Y. \tag{A.8}$$

By Lemma 12.3 each function $f_j(y)$ satisfies equation 12.3 (ii) for $u \in (\varepsilon - I)Y$, $v \in Y$. Taking into account that the function $f_j(y)$ is continuous, it follows from (A.8) that $f_j(y)$ satisfies equation 12.3 (ii) for all $u, v \in Y$. Note now that $X \neq X_{(2)}$ because if $X = X_{(2)}$, then by Theorem 1.11.5 the group X is topologically isomorphic to the group

$$\mathbb{Z}(2)^{\mathfrak{n}} \times \mathbb{Z}(2)^{\mathfrak{m}*}, \tag{A.9}$$

where $\mathfrak{n}$ and $\mathfrak{m}$ are cardinal numbers, the group $\mathbb{Z}(p)^{\mathfrak{n}}$ is considered in the product topology and the group $\mathbb{Z}(p)^{\mathfrak{m}*}$ is considered in the discrete topology. Since $X \not\approx \mathbb{Z}(2)$ and X is a group of the form (A.9), X is not regularly complete. We deduce from $X \neq X_{(2)}$ that $-I \neq I$. Since the group X is regularly complete, $-I$ is a regular automorphism of the group X. Hence $X_{(2)} = \{0\}$. Applying Proposition A.4, we obtain the desired representation for the function $f_j(y)$. □

Let $\varepsilon \in \mathrm{Aut}(Y)$, $\delta = \tilde{\varepsilon}$. Assume that X is not regularly complete. The proof of (a) $\Longrightarrow$ (b) in Theorem A.11 shows that the condition of regularity of δ is necessary in order that all non-vanishing continuous normalized Hermitian solutions of equation (A.6) can be represented in the form A.11 (i). Let $\delta \in \mathrm{Aut}(X)$ be a regular automorphism. The following natural question arises: For which groups Y does the regularity of δ imply that all non-vanishing continuous normalized Hermitian solutions of equation (A.6) can be represented in the form A.11 (i)? Obviously, to solve this problem we should restrict ourselves to such groups Y for which there exist regular automorphisms of the group X. In particular for such groups the condition $X_{(2)} \not\approx \mathbb{Z}(2)$ holds. We need some lemmas.

Lemma A.12. *Assume that $\varepsilon \in \mathrm{Aut}(Y)$, and $\delta = \tilde{\varepsilon}$ is a regular automorphism. Then all non-vanishing continuous normalized Hermitian functions $f_j(y)$ satisfying equation (A.6) can be represented in the form*

$$\text{(i)} \quad f_j(y) = m_j(y)(x_j, y) \exp\{\varphi_j(y)\},$$

where $m_j(y)$ are $\overline{Y^{(2)}}$-invariant continuous normalized Hermitian functions satisfying equation (A.6) and taking values ± 1, $x_j \in X$, and $\varphi_j(y)$ are continuous real-valued functions satisfying equation 2.16 (ii).

Proof. Set $\varphi_j(y) = \ln|f_j(y)|$. Taking into account that δ is a regular automorphism and arguing as in the proof of (b) $\Longrightarrow$ (a) in Theorem A.11, we obtain that each function $f_j(y)$ satisfies equation 12.3 (ii) for all $u, v \in Y$. Taking into account that the functions $f_j(y)$ are Hermitian, it follows from equation 12.3 (ii) that the function $\varphi_j(y)$ satisfies equation 2.16 (ii). It is obvious that the function $\varphi_j(y)$ is continuous.

Put $l_j(y) = f_j(y)/|f_j(y)|$. It is obvious that the function $l_j(y)$ satisfies the conditions of Corollary 9.12. Applying Corollary 9.12 we obtain the required assertion. □

Let $Y = (\mathbb{Z}(2))^n$ and $\varepsilon \in \mathrm{Aut}(Y)$. We introduce the following notation. For a given $y \in Y$ denote by $|y|$ the natural k such that the elements $y, \varepsilon y, \dots, \varepsilon^{k-1} y$ are independent, and $\varepsilon^k y \in \langle y, \varepsilon y, \dots, \varepsilon^{k-1} y\rangle$.

Lemma A.13. *Let $Y = (\mathbb{Z}(2))^n$, where $n = 2, 3, 5$. Let ε be a regular automorphism of the group Y. Then there exists an element $\zeta \in Y$ such that $|\zeta| = n$.*

Proof. Note first that there is no subgroup A of Y such that $A \cong (\mathbb{Z}(2))^{n-1}$, and A is invariant with respect to ε. To prove this, assume that such a subgroup A exists. Take $y_0 \notin A$. Then $\varepsilon y_0 = y_0 + a$, where $a \in A$ and $a \neq 0$. Since ε is a regular automorphism and Y is a finite group, we have $\varepsilon - I \in \mathrm{Aut}(Y)$. On the other hand, $(\varepsilon - I)y_0 = a \in A$, but this is impossible because the restriction of the automorphism $\varepsilon - I$ to the subgroup A is an automorphism of the subgroup A. This remark implies the existence of the desired element $\zeta \in Y$ in the cases when $Y = (\mathbb{Z}(2))^2$ and $Y = (\mathbb{Z}(2))^3$.

Let $Y = (\mathbb{Z}(2))^5$ and suppose that $|y| < 5$ for any $y \in Y$. First assume that there exists an element $y_0 \in Y$ such that $|y_0| = 4$. Consider the subgroup $A = \langle y_0, \varepsilon y_0, \varepsilon^2 y_0, \varepsilon^3 y_0\rangle$. Then $A \cong (\mathbb{Z}(2))^4$, and A is invariant with respect to ε. As has been noted above, this is impossible. Hence $|y| \le 3$ for all $y \in Y$. Assume now that $|y_1| = 3$ for some $y_1 \in Y$. In this case the elements y_1, $y_2 = \varepsilon y_1$, and $y_3 = \varepsilon^2 y_1$ are independent and $\varepsilon y_3 \in \langle y_1, y_2, y_3\rangle = Y_1$. Take an element $y_4 \notin Y_1$ and put $y_5 = \varepsilon y_4$. It is obvious that $y_5 \notin Y_1$. Assume first that $\varepsilon y_5 \notin \langle y_4, y_5\rangle$. Then $|y_4| = 3$. Set $Y_2 = \langle y_4, y_5, \varepsilon y_5\rangle$. The subgroup Y_2 by construction is invariant with respect to ε. Hence the subgroup $Y_3 = Y_1 \cap Y_2$ is also invariant with respect to ε. Since $Y_3 \cong \mathbb{Z}(2)$, the invariance of Y_3 with respect to ε contradicts the regularity of ε. Hence $\varepsilon y_5 \in \langle y_4, y_5\rangle$, and this implies that $\varepsilon y_5 = y_4 + y_5$, because if $\varepsilon y_5 = y_4$, then $\varepsilon(y_4 + y_5) = y_4 + y_5$, contrary to the regularity of ε. Note now that if $\varepsilon y_3 = y_1 + y_2 + y_3$, then $\varepsilon(y_1 + y_3) = y_1 + y_3$, contrary to the regularity of ε. This means that either $\varepsilon y_3 = y_1 + y_2$ or $\varepsilon y_3 = y_1 + y_3$. Putting $\tilde{y} = y_1 + y_4$, it is easily seen that in both cases $|\tilde{y}| = 5$, contrary to the assumption. Therefore $|y| \leq 2$ for all $y \in Y$. So, there exist elements $z_j \in Y$, $j = 1, 2, 3, 4$, such that $\varepsilon z_1 = z_2$, $\varepsilon z_2 = z_1 + z_2$, $\varepsilon z_3 = z_4$, $\varepsilon z_4 = z_3 + z_4$. Hence the subgroup $A = \langle z_1, z_2, z_3, z_4 \rangle \cong (\mathbb{Z}(2))^4$ and A is invariant with respect to ε, but as has been noted above, this is impossible. □

Lemma A.14. *Let $Y = (\mathbb{Z}(2))^n$, where $n \geq 2$, and let $e_1, \dots, e_n$ be generators of the group Y. Let ε be an automorphism of Y such that $\varepsilon e_i = e_{i+1}$, $i = 1, 2, \dots, n-1$. If functions $f_j(y)$ are normalized, satisfy equation (A.6), and take values ± 1, then $f_j(y)$ are characters of the group Y.*

Proof. It is obvious that there exist elements $x_1, x_2 \in X$ such that $(x_1, e_i) = f_1(e_i)$, $(x_2, e_i) = f_2(e_i)$, $i = 1, 2, \dots, n$. Since the characters (x_1, y), (x_2, y) satisfy equation (A.6), we can consider the new functions $\tilde{f}_1(y) = f_1(y)(x_1, y)$, $\tilde{f}_2(y) = f_2(y)(x_2, y)$ which also satisfy equation (A.6). These functions satisfy the conditions $\tilde{f}_1(e_i) = \tilde{f}_2(e_i) = 1$, $i = 1, 2, \dots, n$. We will verify that $\tilde{f}_1(y) = \tilde{f}_2(y) = 1$ for all $y \in Y$. Thus the lemma will be proved.

Set $A_k = \langle e_1, \dots, e_k \rangle$. We will prove the desired statement by induction on k. For $y \in A_1$ the statement is true. Assume that the statement is true for $y \in A_k$. Take an arbitrary element $y \in A_{k+1} \setminus A_k$. Then $y = z + e_{k+1}$ for some $z \in A_k$. Substitute $u = z$, $v = e_k$ in (A.6). Then the right-hand side of (A.6) is equal to 1, and $\tilde{f}_1(u + v) = 1$ because $u + v \in A_k$. Hence $\tilde{f}_2(u + \varepsilon v) = \tilde{f}_2(y) = 1$. Substitute $u = e_{k+1}$, $v = z$ in (A.6). It follows from $\varepsilon v \in A_{k+1}$ that $\tilde{f}_2(\varepsilon v) = \tilde{f}_2(u + \varepsilon v) = 1$. Since the right-hand side of (A.6) is equal to 1, we have $\tilde{f}_1(u + v) = \tilde{f}_1(y) = 1$. Thus $\tilde{f}_1(y) = \tilde{f}_2(y) = 1$ for $y \in A_{k+1}$. □

Theorem A.15. *Let a group X satisfy the following condition:* (i) *either $X_{(2)} = \{0\}$ or $X_{(2)} \cong (\mathbb{Z}(2))^n$, where $n = 2, 3, 5$. Assume that $\varepsilon \in \mathrm{Aut}(Y)$ and $\delta = \tilde{\varepsilon}$ is a regular automorphism. If non-vanishing continuous normalized Hermitian functions $f_j(y)$ satisfy equation (A.6), then they can be represented in the form A.11* (i).

Proof. Taking into account Lemma A.12, it suffices to show that the functions $m_j(y)$ in representation A.12 (i) are characters of the group Y. If $X_{(2)} = \{0\}$, then by Theorem 1.9.5, $\overline{Y^{(2)}} = Y$. As has been proved in Lemma A.12, $m_j(y) = 1$ for $y \in \overline{Y^{(2)}}$. Therefore in this case the theorem is proved.

Let $X_{(2)} \cong (\mathbb{Z}(2))^n$, where $n = 2, 3, 5$. As has been proved in Lemma A.12, the functions $m_j(y)$ are $\overline{Y^{(2)}}$-invariant. Hence the functions $m_j(y)$ induce some normalized functions $\widetilde{m_j}(y)$ on the factor group $Y/\overline{Y^{(2)}}$ which satisfy equation (A.6) and take the values ± 1. Note that by Theorems 1.9.2 and 1.9.5, $(Y/\overline{Y^{(2)}})^* \cong X_{(2)}$. It follows from (i) that $Y/\overline{Y^{(2)}} \cong (\mathbb{Z}(2))^n$, where $n = 2, 3, 5$. Since $\varepsilon \in \mathrm{Aut}(\overline{Y^{(2)}})$, the automorphism ε induces an automorphism $\hat{\varepsilon}$ on the factor group $Y/\overline{Y^{(2)}}$. The automorphism $\hat{\varepsilon}$ is the automorphism adjoint to the restriction of the regular automorphism δ to the subgroup $X_{(2)}$. Since $X_{(2)}$ is a finite subgroup, $\hat{\varepsilon}$ is also a regular automorphism. By Lemma A.13 there exists an element $\zeta \in Y/\overline{Y^{(2)}}$ such that $|\zeta| = n$. Put $e_i = \hat{\varepsilon}^{i-1}\zeta$, $i = 1, 2, \dots, n$. Then e_i are generators of the group $Y/\overline{Y^{(2)}}$ and $\hat{\varepsilon}e_i = e_{i+1}$, $i = 1, 2, \dots, n-1$. Applying Lemma A.14 to the group $Y/\overline{Y^{(2)}}$, we obtain that the functions $\widetilde{m_j}(y)$ are characters of the factor group $Y/\overline{Y^{(2)}}$. Hence the functions $m_j(y)$ are also characters of the group Y. □

Remark A.16. Condition A.15 (i) is critical for the correctness of Theorem A.15. Let $Y = (\mathbb{Z}(2))^n$, where either $n = 4$ or $n \geq 6$. Then $X = X_{(2)} \cong (\mathbb{Z}(2))^n$ and condition A.15 (i) is not fulfilled. Assume first that $n = 4$. Denote by $y = (a_1, a_2, a_3, a_4)$, where $a_i \in \mathbb{Z}(2)$, elements of Y. Consider $\varepsilon \in \mathrm{Aut}(Y)$ of the form

$$\varepsilon(a_1, a_2, a_3, a_4) = (a_2, a_1 + a_2, a_4, a_3 + a_4). \tag{A.10}$$

Put $\delta = \tilde{\varepsilon}$. Obviously, ε is a regular automorphism. Since Y is a finite group, $\delta = \tilde{\varepsilon}$ is also a regular automorphism. Consider on the group Y the functions of the form

$$f_1(a_1, a_2, a_3, a_4) = \exp\{\pi i(a_1a_3 + a_1a_4 + a_2a_3\},$$
$$f_2(a_1, a_2, a_3, a_4) = \exp\{\pi i(a_1a_4 + a_2a_3 + a_2a_4\}.$$

It is obvious that the functions $f_j(y)$ do not vanish, and they are normalized and Hermitian. It is easily seen that $f_j(y)$ satisfy equation (A.6). Since the functions $f_j(y)$ are not characters of the group Y, they can not be represented in the form A.11 (i). Thus for the group $Y = (\mathbb{Z}(2))^4$ there exists the automorphism $\varepsilon \in \mathrm{Aut}(Y)$ such that:

(a) $\delta = \tilde{\varepsilon}$ is a regular automorphism;

(b) there exist non-vanishing normalized Hermitian solutions of equation (A.6) such that they can not be represented in the form A.11 (i).

Let $n \geq 6$. Represent the group Y in the form $Y = Y_1 \times Y_2$, where $Y_1 = (\mathbb{Z}(2))^4$, $Y_2 = (\mathbb{Z}(2))^k$, $k \geq 2$. Put $X_j = Y_j^*$, $j = 1, 2$. Note that there exists an automorphism $\alpha \in \mathrm{Aut}(Y_2)$ such that $\tilde{\alpha}$ is a regular automorphism of X_2. The standard reasoning shows that the existence of $\varepsilon \in \mathrm{Aut}(Y)$ such that statements (a) and (b) hold follows from the existence of such ε in the case of $n = 4$.

Note that if $Y = (\mathbb{Z}(2))^n$, $n \geq 2$, and we suppose that $\varepsilon e_n = e_1 + e_n$ in Lemma A.14, then ε is a regular automorphism. Hence $\delta = \tilde{\varepsilon}$ is also a regular automorphism. We conclude now from Lemmas A.12 and A.14 that there exists an automorphism $\varepsilon \in \mathrm{Aut}((\mathbb{Z}(2))^n)$ such that all non-vanishing normalized Hermitian functions $f_j(y)$ satisfying equation (A.6) are of the form A.11 (i).

In connection with Remark A.16 we will prove the following statement.

Proposition A.17. *There exists a group Y which has the properties:*

(a) *there exists an automorphism $\varepsilon \in \mathrm{Aut}(Y)$ such that $\delta = \tilde{\varepsilon}$ is a regular automorphism of the group X;*

(b) *for any* $\varepsilon \in \mathrm{Aut}(Y)$ *there exist non-vanishing continuous normalized Hermitian solutions of equation* (A.6) *such that they can not be represented in the form* A.11 (i).

Proof. Following [51], § 116, Example 4), take two disjoint sets of prime numbers $\{p_i\}$ and $\{q_i\}$ such that $p_i \equiv 1 \pmod 4$ and $q_i \equiv 1 \pmod 6$. Take integers k_i and l_i which satisfy the conditions $k_i^2 \equiv 1 \pmod{p_i}$ and $l_i^2 + l_i \equiv -1 \pmod{q_i}$. Put

$$Y = \langle a, b, c, d,\ p_i^{-1}(a + k_i b),\ p_i^{-1}(c + k_i d),\ q_i^{-1}(a + l_i c),\ q_i^{-1}(d + l_i b) \text{ for all } i \rangle.$$

The group Y is countable. Consider Y with the discrete topology. As has been shown in [51], § 116, the group $\mathrm{Aut}(Y)$ consists of the following automorphisms:

$$\begin{aligned}
&\varepsilon_1 = I;\\
&\varepsilon_2 = -I;\\
&\varepsilon_3\colon a \mapsto b,\ b \mapsto -a,\ c \mapsto b - d,\ d \mapsto c - a;\\
&\varepsilon_4 = -\varepsilon_3;\\
&\varepsilon_5\colon a \mapsto c,\ b \mapsto d,\ c \mapsto c - a,\ d \mapsto d - b;\\
&\varepsilon_6 = -\varepsilon_5;\\
&\varepsilon_7\colon a \mapsto d,\ b \mapsto -c,\ c \mapsto b,\ d \mapsto -a;\\
&\varepsilon_8 = -\varepsilon_7;\\
&\varepsilon_9\colon a \mapsto a - c,\ b \mapsto b - d,\ c \mapsto a,\ d \mapsto b;\\
&\varepsilon_{10} = -\varepsilon_9;\\
&\varepsilon_{11}\colon a \mapsto b - d,\ b \mapsto c - a,\ c \mapsto -d,\ d \mapsto c;\\
&\varepsilon_{12} = -\varepsilon_{11}.
\end{aligned}$$

Since Y is a torsion-free group of finite rank, it follows from 1.13 (b) that for any $\varepsilon \in \mathrm{Aut}(Y)$ the adjoint automorphism $\delta = \tilde{\varepsilon}$ is regular if and only if $(\varepsilon - I) \in \mathrm{Aut}(Y)$. This implies that there exist only two regular topological automorphisms of the group X, namely $\delta_5 = \tilde{\varepsilon}_5$ and $\delta_9 = \tilde{\varepsilon}_9$. Thus statement (a) is proved. Let us prove (b). We conclude from the proof of $(a) \Rightarrow (b)$ in Theorem A.11 that if $\varepsilon \in \mathrm{Aut}(Y)$ and $\delta = \tilde{\varepsilon}$ is not a regular automorphism, then there exist non-vanishing continuous normalized Hermitian solutions of equation (A.6) such that they can not be represented in the form A.11 (i). It still remains to prove that the automorphisms ε_5 and ε_9 for which δ_5 and δ_9 are regular have the same property. Denote by $\hat{\varepsilon}_5$, $\hat{\varepsilon}_9$ the automorphisms on the factor group $Y/Y^{(2)}$ induced by ε_5, ε_9. It is obvious that $Y/Y^{(2)} \cong (\mathbb{Z}(2))^4$. It is easily verified that the automorphisms $\hat{\varepsilon}_5$, $\hat{\varepsilon}_9$ operate on the factor group $Y/Y^{(2)}$ under appropriate choice of generators in $Y/Y^{(2)}$ by formula (A.10). Hence as has been noted in Remark A.16, for $\hat{\varepsilon}_5$ and $\hat{\varepsilon}_9$ there exist non-vanishing normalized Hermitian solutions of equation (A.6) which take values ± 1 such that they can not be represented in the form A.11 (i). These solutions one can consider as functions on Y which have the same properties with respect to the automorphisms ε_5 and ε_9. □

A.18. We will study now non-vanishing measurable normalized Hermitian solutions of equation (A.6). Obviously, any automorphism $\varepsilon \in \mathrm{Aut}(Y)$ can be regarded as an automorphism of the group Y_d. Let $\delta \in \mathrm{Aut}(X)$. Consider the adjoint automorphism $\tilde{\delta} \in \mathrm{Aut}(Y)$ as an automorphism of the group Y_d. Since $bX \cong (Y_d)^*$, denote by $\bar{\delta}$ the topological automorphism of the group bX adjoint to $\tilde{\delta} \in \mathrm{Aut}(Y_d)$. The group X is a dense subgroup of the group bX, and it is easily seen that the restriction of the automorphism $\bar{\delta}$ to the subgroup X coincides with δ.

An automorphism $\delta \in \mathrm{Aut}(X)$ is called *strongly regular*, if the automorphism $\bar{\delta} \in \mathrm{Aut}(bX)$ is regular.

Example A.19. We give an example of a group X and a regular automorphism $\delta \in \mathrm{Aut}(X)$ which is not strongly regular. For this purpose we slightly modify the construction from [59], (24.44). Denote by Y a set of sequences (a_k), where a_k is a 2^pth root of unity for some natural p, and $a_k = \pm 1$ for all but a finite set of indices. Obviously, Y is an Abelian group with respect to multiplication. For a finite set Λ of indices k let U_Λ be the set of all (a_k) such that $a_k = 1$ for $k \in \Lambda$ and $a_k = \pm 1$ for $k \notin \Lambda$. Let all U_Λ be taken as an open basis at the identity (1_k). With respect to the introduced topology, Y is a second countable locally compact Abelian group. Note that $Y^{(2)} \neq Y$ and $\overline{Y^{(2)}} = Y$. Put $X = Y^*$, and let $\delta \in \mathrm{Aut}(X)$ be of the form $\delta x = 3x$, $x \in X$. Set $\varepsilon = \tilde{\delta}$. Then $\varepsilon y = 3y$, $y \in Y$ and $\bar{\delta} x = 3x$, $x \in bX$. Since the image $(\varepsilon - I)Y = Y^{(2)}$ is dense in Y, we conclude from 1.13 (b) that $X_{(2)} = \{0\}$, and hence δ is a regular automorphism. Since $(bX)^* \cong Y_d$, it follows from Theorems 1.9.2 and 1.9.5 that $((bX)_{(2)})^* \cong Y_d/(Y_d)^{(2)}$. Therefore $\bar{\delta} \in \mathrm{Aut}(bX)$ is not a regular automorphism. Hence δ is not strongly regular.

A group X is called *strongly regularly complete* if all its topological automorphisms except the identity automorphism are strongly regular.

Theorem A.20. *Assume that a group Y is not topologically isomorphic to the group $\mathbb{Z}(2)$. Let ε be an arbitrary topological automorphism of Y, $\varepsilon \neq I$. The following statements are equivalent:*

(a) *all non-vanishing measurable normalized Hermitian functions $f_j(y)$ on Y satisfying equation (A.6) are represented in the form A.11* (i);

(b) *X is strongly regularly complete.*

Proof. (a) $\Longrightarrow$ (b). Assume that X is not strongly regularly complete. We will prove that there exists an automorphism $\varepsilon \in \mathrm{Aut}(Y)$, $\varepsilon \neq I$, and non-vanishing measurable normalized Hermitian functions $f_j(y)$ on the group Y satisfying equation (A.6) and such that they can not be represented in the form A.11 (i). By the condition there exists a not strongly regular automorphism $\delta \in \mathrm{Aut}(X)$. Put $\varepsilon = \tilde{\delta}$, $H = (\varepsilon - I)Y$. Since δ is not strongly regular, we have $H \neq Y$. Let $a \in \mathbb{R}$, $a \neq 0$, $a \neq \pm 1$. Consider on the group Y the functions

$$f_1(y) = \begin{cases} 1 & \text{if } y \in H, \\ a & \text{if } y \notin H, \end{cases} \qquad f_2(y) = \begin{cases} 1 & \text{if } y \in H, \\ 1/a & \text{if } y \notin H. \end{cases}$$

Obviously, $f_j(y)$ are non-vanishing measurable normalized Hermitian functions. We will verify that the functions $f_j(y)$ satisfy equation (A.6). Since $\varepsilon H = H$, the right-hand side of equation (A.6) is equal to 1 for all $u, v \in Y$. Suppose that there exist $u, v \in Y$ such that the left-hand side of equation (A.6) is not equal to 1. This implies that either $u + v \in H, u + \varepsilon v \notin H$ or $u + v \notin H, u + \varepsilon v \in H$. In both cases we have $(\varepsilon - I)v \notin H$ contrary to the definition of H. Thus the functions $f_j(y)$ satisfy equation (A.6) and obviously, they can not be represented in the form A.11 (i).

(b) $\Longrightarrow$ (a). Reasoning as in the proof of (b) $\Longrightarrow$ (a) in Theorem A.11 we obtain that the strongly regularity of the group X implies that $(\varepsilon - I)Y = Y$ for any $\varepsilon \in \mathrm{Aut}(Y)$, $\varepsilon \neq I$. Moreover $(bX)_{(2)} = \{0\}$. The statement of the theorem follows now from Lemma 12.3 and Proposition A.8. □

Let $\varepsilon \in \mathrm{Aut}(Y)$ and $\delta = \tilde{\varepsilon}$. Assume that X is not strongly regularly complete. The proof of $(a) \Rightarrow (b)$ in Theorem A.19 shows that the condition of strong regularity of δ is necessary in order that all non-vanishing measurable normalized Hermitian solutions of equation (A.6) can be represented in the form A.11 (i). Let $\delta \in \mathrm{Aut}(X)$ be a strongly regular automorphism. The following natural question arises: For which groups Y can the strong regularity of δ imply that all non-vanishing measurable normalized Hermitian solutions of equation (A.6) be represented in the form A.11 (i)? Obviously, to solve this problem we should restrict ourselves to such groups Y for which there exist strongly regular automorphisms of the group X. In particular for such groups the condition $(bX)_{(2)} \ncong \mathbb{Z}(2)$ holds.

Theorem A.21. *Let a group X satisfy the condition:* (i) *either $(bX)_{(2)} = \{0\}$ or $(bX)_{(2)} \cong (\mathbb{Z}(2))^n$, where $n = 2, 3, 5$. Assume that $\varepsilon \in \mathrm{Aut}(Y)$ and $\delta = \tilde{\varepsilon}$ is a strongly regular automorphism. If non-vanishing measurable normalized Hermitian functions $f_j(y)$ satisfy equation* (A.6)*, then they can be represented in the form* A.11 (i).

Proof. Put $\varphi_j(y) = \ln|f_j(y)|$. Arguing as in the proof of (b) $\Longrightarrow$ (a) in Theorem A.11 we obtain that if δ is a strongly regular automorphism, then $Y = (\varepsilon - I)Y$. Applying Lemma 12.3 we get that the functions $f_j(y)$ satisfy equation 12.3 (ii) for all $u, v \in Y$. Inasmuch as $f_j(y)$ are normalized Hermitian functions, each of the functions $\varphi_j(y)$ satisfies equation 2.14 (ii). Since the functions $\varphi_j(y)$ are measurable, by Lemma A.5 the functions $\varphi_j(y)$ are continuous.

Arguing as in the proof of (b) $\Longrightarrow$ (a) in Proposition A.8 we make sure that condition (i) implies that the subgroup $Y^{(2)}$ is closed. In the final part of the proof we reason as in the proof of Theorem A.15. □

Comments and unsolved problems

Chapter II

The standard definition of a Gaussian distribution on a locally compact Abelian group was first introduced by Parthasarathy, Ranga Rao, and Varadhan in [90]. They define a Gaussian distribution γ as a distribution satisfying the conditions of Proposition 3.16 and prove that γ is Gaussian if and only if its characteristic function can be represented in the form 3.1 (i). In [90] the main properties of the Gaussian distributions were also studied. We note that in [90] along with other results the representation of the characteristic function of an infinitely divisible distribution on a locally compact Abelian group (the Lévy–Khinchin formula) was obtained. The results of the article [90] were included in the monographs [63] and [89].

Propositions 3.6, 3.16, 3.17, and Remarks 3.3, 3.4, 3.12 belong to Parthasarathy, Ranga Rao, and Varadhan ([90]). We follow [63] in the proof of Lemma 3.18 and in the presentation of items 3.15 and 3.19 (b). We follow the review [95] by Sazonov and Tutubalin in the proof of Proposition 3.6. We also note that in [63] a part of the results of the article [90] was generalized on locally compact Abelian groups which need not be second countable.

The Gaussian distributions on the finite-dimensional torus $\mathbb{T}^n$ were studied by Siebert in [97] (see also [63], § 5.5). The Gaussian distributions on the infinite-dimensional torus $\mathbb{T}^{\aleph_0}$ which correspond to diagonal matrices in Proposition 3.11 were studied by Bendikov in [3], [4], Siebert in [97], and Berg in [11] (see also [5], [7], [12]). Propositions 3.8, 3.11, and 3.14 were proved by Feldman in [27]. In connection with Propositions 3.11 and 3.14 see also the article [8] by Bendikov and Saloff-Coste. We note that Siebert in [97] proved that if on a connected locally compact Abelian group X an absolutely continuous Gaussian distribution exists, then X must be locally connected and second countable. The existence of absolutely continuous Gaussian distributions on such groups was proved independently by Bendikov ([3]), Siebert ([97]), and Berg ([11]). Statement 3.19 (c) belongs to Feldman. Gaussian semigroups on infinite-dimensional locally compact groups, not necessarily Abelian, were studied recently by Bendikov and by Bendikov and Saloff-Coste in connection with the theory of potential on these groups constructed by them (see e.g. the monograph [6] by Bendikov, and see also the articles [9] and [10] where one can find further references).

The classical Cramér theorem on decomposition of the Gaussian distribution on the real line was proved in [21]. It was the first result in arithmetic of probability distributions. It is interesting to remark that all known proofs of the Cramér theorem are based on the application of complex analysis. The monographs [22], [73], [74] and reviews [75], [88] are devoted to arithmetic of probability distributions on the real line and in the space $\mathbb{R}^m$. The main part of the monographs [76] and [92] is devoted to the same subject. The basic results on arithmetic of probability distributions on locally compact Abelian groups are contained in [33].

Marcinkiewicz noted in [77] that any Gaussian distribution on the circle group $\mathbb{T}$ has non-Gaussian factors. This result was re-proved by Martin-Löf (in [57], Remark 4.5.1) and by Carnal ([18]). Decomposition 4.1 (i) was mentioned by Heyer in [62]. Feldman in [26] proved that if X is a compact Abelian group such that $X \ncong \mathbb{Z}(2)$ and a distribution $\gamma \in \mathrm{M}^1(X)$ satisfies condition 4.1 (i), then γ has an indecomposable factor. Lemmas 4.2 and 4.3 were proved by Feldman in [25] and [27]. Lemma 4.4 and Proposition 4.5 were proved by Feldman and Fryntov in [45]. Theorem 4.6 and Proposition 4.9 were proved by Feldman in [25]. In the proof of Theorem 4.6 we follow [27].

There are several different and nonequivalent definitions of polynomials on Abelian groups (see e.g. [71], where these definitions are compared and some algebraic conditions on a group are determined when these definitions are equivalent). Results of 5.1–5.5 belong to Djoković ([24]). Lemma 5.10 was proved by Feldman in [25]. Statements 5.11–5.14 belong to Feldman ([30]).

Urbanik in [100] defines a Gaussian distribution on a locally compact Abelian group as a distribution γ which can be included into a continuous one-parameter subgroup $(\gamma_t)_{t\geq 0}$, $\gamma_0 = E_0$, $\gamma_1 = \gamma$, and satisfying condition 3.17 (ii). An example of a non-Gaussian distribution on the two-dimensional torus $\mathbb{T}^2$ satisfying condition 3.17 (ii) was constructed in [100]. Lemmas 6.3, 6.4, and Theorem 6.6 belong to Feldman ([28]).

Problems. II.1. In view of Proposition 3.14, Gaussian distributions on connected and not locally connected locally compact Abelian groups are singular. If X is a connected and locally connected locally compact Abelian group of finite dimension, i.e., $X \cong \mathbb{R}^m \times \mathbb{T}^n$, where $m \geq 0, n \geq 0$, then the structure of a Gaussian distribution on X is well known. It is either absolutely continuous or singular (see 3.15). The structure of a Gaussian distribution on connected and locally connected locally compact Abelian groups X of infinite dimension is unknown, i.e., when $X \cong \mathbb{R}^m \times \mathbb{T}^{\aleph_0}$, $m \geq 0$.

II.2. It is well known that two Gaussian distributions in the space $\mathbb{R}^{\aleph_0}$ are either mutually absolutely continuous or mutually singular ([58]). Sazonov and Tutubalin in [95] formulated the problem: to prove or disprove the analogous statement for Gaussian distributions on locally compact Abelian groups. In [27] Feldman solved this problem for connected locally compact Abelian groups of finite dimension, and for connected locally compact Abelian groups of infinite dimension containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. The Sazonov–Tutubalin problem for connected locally compact Abelian groups of infinite dimension containing a subgroup topologically isomorphic to the circle group $\mathbb{T}$ is still unsolved. We note that the positive solution of this problem implies the solution of Problem II.1.

Chapter III

The statement known as the Kac–Bernstein theorem (the independence of the sum and the difference of two independent random variables implies that the random variables are Gaussian) was discovered independently by Kac in [65] and by Bernstein in [13]. This result is one of the most celebrated characterization theorems. A large number of publications are devoted to characterization problems of mathematical statistics. In particular we mention the fundamental monograph [66] by Kagan, Linnik, and Rao.

In the case when random variables take values in a locally compact Abelian group X, some analogues of the Kac–Bernstein theorem were considered by Rukhin in [93], [94] and by Heyer and Rall in [64]. Rukhin proved in [93] the following statement: Let X and Y be Corwin groups, ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2. If $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent, then $\mu_j \in \Gamma(X) * I(X)$ (compare with Theorem 7.10). We note that the Rukhin paper [93] was the first article where a classical characterization theorem was studied on arbitrary locally compact Abelian groups. Proposition 7.4 was proved by Rukhin ([93]) and by Heyer and Rall ([64]). Lemmas 7.5, 7.8, 7.9, and Theorem 7.10 were proved by Feldman in [29]. Lemma 7.6 was proved by Feldman in [32]. Lemma 7.7 was proved by Feldman in [33], Proposition 9.20. Results of 7.13–7.16 belong to Myronyuk ([78]).

Let ξ_1 and ξ_2 be independent random variables with values in a locally compact Abelian group X and distributions μ_1 and μ_2 such that their sum and the difference are independent. If the connected component of zero of the group X contains no elements of order 2, then by Theorem 7.10 the distributions μ_j are convolutions of Gaussian and idempotent distributions. The article [2] by Baryshnikov, Eisenberg, and Stadje was the first research in which distributions μ_j were described in the case when the connected component of zero of a group X contains elements of order 2. They described the distributions μ_j in the case when $X = \mathbb{T}$. Lemma 8.2 was proved in [2]. The proof of Lemma 8.2 given here belongs to Myronyuk ([80]). Theorems 8.5 and 8.8 were proved by Myronyuk in [80].

Lemmas 9.4, 9.6, and 9.7 were proved by Feldman in [29]. Lemma 9.5 was proved by Feldman in [39]. The main result of Section 9, i.e., the complete description of groups X for which equality (9.2) holds (Theorem 9.9) was proved by Feldman in [29] and [39]. It should be noted that Rukhin in [93] proved equality (9.2) assuming that Y is a Corwin group. Heyer and Rall in [64] proved equality (9.2) for groups X satisfying the following conditions:

(i) X is a Corwin group;

(ii) either $Y = \overline{Y^{(2)}}$ or $Y/\overline{Y^{(2)}} \cong \mathbb{Z}(2)$.

Proposition 9.8 and Lemma 9.10 belong to Feldman. Taking into account Lemmas 7.5 and 9.10 it is easy to see that the above mentioned results of Rukhin and Heyer–Rall follow from Theorem 9.9. Results 9.11–9.14 belong to Myronyuk ([78]).

We note that Gaussian distributions in the sense of Bernstein on arbitrary topological separable metric Abelian groups were studied by Byczkowski in [16], [17] and by Byczkowska and Byczkowski in [15]. In particular, for a wide class of such groups X, in [16], [17] it was proved that if μ is a Gaussian distributions in the sense of Bernstein on X without idempotent factors and B is a Borel subgroup of X, then either $\mu(G) = 0$ or $\mu(G) = 1$ (the zero-one low).

We make some remarks on non-Abelian groups. Let X be a second countable non-Abelian topological group, not necessarily locally compact. The operation in X we write as multiplication. Let ξ_1 and ξ_2 be independent identically distributed random variables with values in X and distribution μ. In the article [85] Neuenschwander suggested in the non-Abelian case, instead of the condition

(i) $\xi_1\xi_2$ and $\xi_1\xi_2^{-1}$ are independent,

considering the condition

(ii) $(\xi_1\xi_2, \xi_2\xi_1)$ and $(\xi_1\xi_2^{-1}, \xi_2^{-1}\xi_1)$ are independent.

We note that if X is an Abelian group, then conditions (i) and (ii) are equivalent. In the case when X is the Heisenberg group $X = \mathscr{H}$, Neuenschwander proved that it follows from condition (ii) that μ is a Gaussian distribution concentrated on an Abelian subgroup of $\mathscr{H}$ ([85]). Then Neuenschwander, Roynette, and Schott generalized this result to simply connected nilpotent Lie groups ([86]). Graczyk and Loeb proved in [56] that if X is an arbitrary nilpotent group, then condition (ii) implies that the distribution μ is concentrated on an Abelian subgroup of X. In contrast to [86] the proof by Graczyk–Loeb uses neither the fact that X is a Lie group nor the Campbell–Hausdorff formula. In [56] they also proved that if X is either a discrete group or a compact group, or X is in a class of solvable groups (e.g. the "$ax + b$" group), then condition (ii) implies that the distribution μ is concentrated on an Abelian subgroup of X. As a corollary they show that the Gaussian measures on Riemannian symmetric spaces do not satisfy the property (ii).

Problems. III.1. Let ξ_1 and ξ_2 be independent random variables with values in a connected locally compact Abelian group X of dimension l and distributions μ_1 and μ_2. Assume that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. Describe possible distributions μ_j. We note that if $l = 1$, then either $X = \mathbb{R}$ or $X = \mathbb{T}$ or $X = \Sigma_{\boldsymbol{a}}$, and the description of possible distributions μ_j follows from the Kac–Bernstein theorem, Corollary 8.6, and Theorem 8.8 respectively.

III.2. Problem III.1 under assumption that X is a connected locally compact Abelian group of infinite dimension.

III.3. Problem III.1 for an arbitrary locally compact Abelian group X. Taking into account Lemma 7.5, this problem is reduced to the case when $X = \mathbb{R}^m \times K$, $m \geq 0$ and K is a compact Corwin group.

III.4. To describe supports of Gaussian distributions in the sense of Bernstein. Taking into account Lemma 7.5 one can assume that supports are cosets of subgroups G of the form $G \cong \mathbb{R}^m \times K$, where $m \geq 0$, and K is a Corwin group.

Chapter IV

Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables and α_j, β_j be nonzero real numbers. The theorem stating that the independence of the linear forms $L_1 = \alpha_1\xi_1 + \dots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \dots + \beta_n\xi_n$ implies that all ξ_j are Gaussian was proved independently by Skitovich in [98] and by Darmois in [23]. This result summed up a series of researches generalizing the Kac–Bernstein theorem. The Skitovich–Darmois theorem was extended by Ghurye and Olkin in [54] for the case when instead of random variables, random vectors ξ_j in the space $\mathbb{R}^m$ are considered, and coefficients of the linear forms L_1 and L_2 are nonsingular matrices. Also, in this case the independence of the linear forms implies that all random vectors ξ_j are Gaussian.

A group analogue of the Skitovich–Darmois theorem was studied first by Feldman in [31]. It was assumed in [31] that the independent random variables take values in a locally compact Abelian group, coefficients of the linear forms are integers, and the characteristic functions of independent random variables do not vanish. A group analogue of the Skitovich–Darmois theorem in the case when coefficients of the linear forms are topological automorphisms of the group was considered first by Feldman in [34].

Theorem 10.3 was proved by Feldman in [40]. We note that Theorem 10.3 for an arbitrary $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ was proved in [34]. Remark 10.5 was also established in [40]. Proposition 10.7 was proved by Feldman in [31]. The proof of Proposition 10.7 given here belongs to Myronyuk. Theorem 10.11 belongs to Schmidt ([96]). The Schmidt proof is based on the Lévy–Khinchin formula 2.16 (i). The proof given here belongs to Feldman. It differs from the Schmidt proof and does not use the Lévy–Khinchin formula. Theorem 10.15, Propositions 10.16 and 10.17 belong to Feldman ([40]). Formulated in Remark 10.18, the equivalence of statement (α) in Remark 10.18 and the Cramér theorem on decomposition of the Gaussian distribution belongs to Linnik ([72]), who proved it for the real line. The Linnik proof can be extended without changes to locally compact Abelian groups. Results of Section 11 belong to Myronyuk and Feldman ([82], [84]). Results of Section 12 also belong to Myronyuk and Feldman. We note that Lemma 12.3 essentially belongs to Lajko ([70]), who proved it for the case $Y = \mathbb{R}$. The proof of Lemma 12.3 for locally compact Abelian groups follows the proof of the lemma for the real line.

Problems. IV.1. Describe possible distributions μ_j in Theorem 11.5 in case (IIb) (compare with Theorems 12.4 and 12.9).

IV.2. Let X be a connected locally compact Abelian group of finite dimension l. Let $\delta \in \mathrm{Aut}(X)$ and ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Assume that the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ are independent. Describe possible distributions μ_j.

The complete solution of this problem for the case when $l = 1$ follows from the Skitovich–Darmois theorem, Corollary 8.6, and Theorem 10.3. A partial solution of this problem for the case when $l = 2$ follows from Theorems 10.3, 11.5, and 12.4 (the case of the two-dimensional torus $X = \mathbb{T}^2$ is not fully investigated). The case when $l \geq 3$ and a group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$ has not been investigated at all.

Chapter V

A group analogue of the Skitovich–Darmois theorem, in the case when independent random variables ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, with distributions μ_j take values in a finite Abelian group X and coefficients of the linear forms are automorphisms of X, was considered by Feldman in [34]. Lemmas 14.1 and 14.3 were proved in [34]. From Lemmas 14.1 and 14.3 follows a description of the class of finite Abelian groups for which, for any $n \geq 2$, independence of the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ implies that either all μ_j are degenerate distributions or at least two distributions μ_{j_1} and μ_{j_2} are idempotent. It was also proved in [34] that the obtained description of the class of finite Abelian groups is sharp in the following sense: if a finite Abelian group X does not belong to the class mentioned above, then either for $n = 4$ or for $n = 6$ there exist automorphisms α_j, β_j of X and independent random variables ξ_j with values in X and distributions $\mu_j \notin I(X)$ such that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent. Analogous results were obtained by Feldman for compact Abelian groups ([36]) and for discrete torsion Abelian groups ([37]). Lemma 14.9 was proved in [36]. Theorem 13.5 and Lemmas 13.9, 14.14, 14.15, and 14.17 where proved in [37]. Lemma 14.20 belongs to Myronyuk ([79]); it is a modification of Proposition 1 proved in [34]. Propositions 14.21 and 14.22 also belong to Myronyuk ([79]). They generalize results obtained earlier by Feldman (see [36, Theorem 2] and [37, Theorem 2]). Theorem 14.23 belongs to Myronyuk ([79]).

In the article [38] Feldman showed for finite Abelian groups that the classes of groups for which the Skitovich–Darmois theorem holds for $n = 2$ independent random variables and for $n > 2$ independent random variables are essentially different. Theorems 13.1 and 13.3 were proved by Feldman in [38]. Then Feldman and Graczyk studied the Skitovich–Darmois theorem for two independent random variables on compact totally disconnected Abelian groups and compact connected Abelian groups ([46]). The results of 13.21–13.27 belong to them. The Skitovich–Darmois theorem for two independent random variables on discrete Abelian groups was studied by Feldman and Graczyk in [47]. The results of 13.13–13.18 belong to them. It is interesting to remark that the proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n = 2$ (Theorem 13.17) in contrast to the proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n > 2$ (Theorem 14.19 and Proposition 14.22) does not use the Ulm theory.

The cases when the number of independent random variables $n = 3$ and $n \geq 4$ differ from each other. The Skitovich–Darmois theorem for three independent random variables on finite Abelian groups was studied by Graczyk and Feldman in [55]. The results of 14.4–14.7 belong to them ([55]). Following the scheme of the proof of Theorem 14.7, Myronyuk studied the Skitovich–Darmois theorem for three independent random variables for compact totally disconnected Abelian groups and for discrete torsion Abelian groups ([79]). Lemmas 14.12 and 14.18 belong to her ([79]).

The results of Section 15 belong to Feldman ([44]).

Problems. V.1. Find out if there is a compact Abelian group X such that X is neither connected nor totally disconnected and X has the property: if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j and $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, then independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$?

V.2. Problem V.1 under the additional assumption that $c_X \cong \Sigma_{\boldsymbol{a}}$, where $\boldsymbol{a} = (2, 3, 4, \dots)$.

V.3. Let $X = \mathbb{T}^2$. Is there is an automorphism $\delta \in \mathrm{Aut}(X)$ with the following property: if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$? We note that the reasoning of item 1 in the proof of Theorem 11.5 implies that if $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $ad - bc = 1, a + d = 2$, then μ_1, μ_2 are degenerate distributions.

V.4. On which compact connected Abelian groups X does there exist an automorphism $\delta \in \mathrm{Aut}(X)$ such that, if ξ_1 and ξ_2 are independent random variables with values in X and distributions μ_j, then independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta\xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$?

If $f_2 \in \mathrm{Aut}(X)$, then by Theorem 7.10 the independence of the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$. It follows from Theorem 15.7 that for $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$ the condition $f_2 \in \mathrm{Aut}(X)$ is not only sufficient but necessary for the existence of an automorphism $\delta \in \mathrm{Aut}(X)$ with the property mentioned above. As follows from the example constructed in Remark 15.9 the condition $f_2 \in \mathrm{Aut}(X)$ is not necessary for arbitrary compact connected Abelian groups.

V.5. Find out if there is a locally compact Abelian group X such that X is not topologically isomorphic to a group of the form 14.23 (i), and X has the property: for any independent random variables ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, with values in X and distributions μ_j and for any automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ the independence of the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ implies that all $\mu_j \in \Gamma(X)$.

Chapter VI

The theorem on characterization of a Gaussian distribution on the real line by the symmetry of the conditional distribution of one linear form given another was proved by Heyde in [61].

A group analogue of the Heyde theorem in the case when independent random variables take values in a locally compact Abelian group and coefficients of the linear forms are topological automorphisms of the group was considered first by Feldman in [41] for finite Abelian groups. The results of Section 16 belong to Feldman ([42]). The results of 17.1–17.5 were proved by Feldman in [41]. Proposition 17.7, Lemmas 17.15, 17.16, and Theorem 17.17 belong to Feldman ([41]). The results of 17.8–17.14 belong to Myronyuk and Feldman ([81]). Theorem 17.14 for finite Abelian groups without elements of order 2 was proved in [41]. Proposition 17.6 and statements 17.19–17.27 belong to Feldman ([43]).

Problems. Theorems 16.2 and 16.8 allow us to assume that the following statement holds:

Hypothesis. *Let α_j, β_j be topological automorphisms of a locally compact Abelian group X such that $\beta_1\alpha_1^{-1} \pm \beta_2\alpha_2^{-1} \in \mathrm{Aut}(X)$. Let ξ_1 and ξ_2 be independent random variables with values in X and distributions μ_1 and μ_2 with non-vanishing characteristic functions. Let G be the subgroup of the group X generated by all elements of order* 2*. Assume that the conditional distribution of the linear form $L_2 = \alpha_2\xi_1 + \beta_2\xi_2$ given $L_1 = \alpha_1\xi_1 + \beta_1\xi_2$ is symmetric. Then $\mu_j = \gamma_j * \rho_j$, where $\gamma_j \in \Gamma(X)$, $\sigma(\rho_j) \subset G$, $j = 1, 2$.*

VI.1. Prove the hypothesis for the following groups: $X = \mathbb{T}^3$; $X = \mathbb{T}^n$, $n \geq 4$; $X = \mathbb{T}^{\aleph_0}$; X is an arbitrary connected group; X is an arbitrary group.

VI.2. Prove an analogue of the Heyde theorem for compact totally disconnected Abelian groups (compare with Theorem 13.25).

VI.3. Prove an analogue of the Heyde theorem for $\boldsymbol{a}$-adic solenoids (compare with Theorem 15.7).

Appendix

The results of the appendix belong to Feldman and Myronyuk ([83]).

Let X be a locally compact Abelian group with unique division by 2, i.e., $f_2 \in \mathrm{Aut}(X)$. Let $\tau\colon X^2 \mapsto X^2$ be the mapping defined by $\tau(t,s) = (t+s, t-s)$. Corwin studied complex-valued measures μ_j and ν_j on X satisfying the condition $(\mu_1 \otimes \mu_2)(\tau(E)) = (\nu_1 \otimes \nu_2)(E)$ for any Borel set $E \subset X^2$ ([19], [20]). Under the assumption that the measures μ_j and ν_j are absolutely continuous with respect to a Haar measure m_X, the derivatives $f_j(t)$ and $g_j(t)$ of the measures μ_j and ν_j satisfy the equation

$$f_1(t+s)f_2(t-s) = g_1(t)g_2(s)$$

for almost all $(t,s) \in X^2$. Obviously, the Kac–Bernstein functional equation is a particular case of this equation.

Problems. A.1. Let Y be a locally compact Abelian group which is not second countable. Let $\varepsilon \in \mathrm{Aut}(Y)$. Is the subgroup $(I-\varepsilon)Y$ measurable? In particular, is the subgroup $Y^{(2)}$ measurable?

A.2. If X is a regularly complete second countable locally compact Abelian group, must X be strongly regularly complete?

A.3. Is condition A.15 (i) necessary in order that non-vanishing normalized continuous Hermite functions $f_j(y)$ satisfying equation (A.6) be of the form A.11 (i)?

We make the following remark. Assume that Y is a countable discrete Abelian group with the properties:

(a) $Y/Y^{(2)} \cong (\mathbb{Z}(2))^n$, where either $n = 4$ or $n \geq 6$;

(b) any automorphism $\varepsilon \in \mathrm{Aut}(Y)$ such that $I - \varepsilon \in \mathrm{Aut}(Y)$ satisfies the condition: under suitable choice of generators e_i in the factor group $Y/Y^{(2)}$ the automorphism $\hat{\varepsilon} \in \mathrm{Aut}(Y/Y^{(2)})$ induced by the automorphism ε is of the form $\hat{\varepsilon}e_i = e_{i+1}$, $i = 1, 2, \dots, n-1$.

It follows from Lemmas A.12 and A.14 that if there exists a group Y with properties (a) and (b), then condition A.15 (i) is not necessary.

Bibliography

[1] D. L. Armacost, *The structure of locally compact abelian groups*. Monographs Textbooks Pure Appl. Math. 68, Marcel Dekker, Inc., New York 1981. 6

[2] Y. Baryshnikov, B. Eisenberg, W. Stadje, Independent variables with independent sum and difference: S^1-case. *J. Multivariate Anal.* **45** (1993), 161–170. vi, 239

[3] A. Bendikov, Spatially homogeneous continuous Markov processes on abelian groups, and harmonic structures. *Uspehi Mat. Nauk* **29** (1974), 215–216 (in Russian). 237

[4] A. Bendikov, Functions that are harmonic for a certain class of diffusion processes on a group. *Teor. Veroyatnost. i Primenen.* **20** (1975), 773–784; English transl. *Theory Probab. Appl.* **20** (1976), 759–769. 237

[5] A. Bendikov, Symmetric stable semigroups on the infinite-dimensional torus. *Exposition. Math.* **13** (1995), 39–79. 237

[6] A. Bendikov, *Potential theory on infinite-dimensional abelian groups*. De Gruyter Stud. Math. 21, Walter de Gruyter & Co., Berlin 1995. 237

[7] A. Bendikov and L. Saloff-Coste, Elliptic diffusions on infinite products. *J. Reine Angew. Math.* **493** (1997), 171–220. 237

[8] A. Bendikov and L. Saloff-Coste, On the absolute continuity of Gaussian measures on locally compact groups. *J. Theoret. Probab.* **14** (2001), 887–898. 237

[9] A. Bendikov and L. Saloff-Coste, Central Gaussian convolution semigroups on compact groups: a survey. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6** (2003), 629–659. 237

[10] A. Bendikov and L. Saloff-Coste, On the sample paths of diagonal Brownian motions on the infinite dimensional torus. *Ann. Inst. H. Poincaré Probab. Statist.* **40** (2004), 227–254. 237

[11] Ch. Berg, Fonctions harmoniques sur T^∞. *C. R. Acad. Sci. Paris Sér. A* **280** (1975), 1519–1521. 237

[12] Ch. Berg, Potential theory on the infinite dimensional torus. *Invent. Math.* **32** (1976), 49–100. 237

[13] S. N. Bernstein, On a property which characterizes a Gaussian distribution. *Proc. of Leningrad Polytechnic Institute* **217** (1941), 21–22 (in Russian). v, 239

[14] N. Bourbaki, *General topology. Chapters 5–10*, Springer-Verlag, Berlin 1989. 33

[15] H. Byczkowska and T. Byczkowski, A generalization of the zero-one law for Gaussian measures on metric abelian groups. *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **29** (1981), 187–191. 239

[16] T. Byczkowski, Gaussian measures on metric abelian groups. In *Probability measures on groups* (Proc. Fifth Conf., Oberwolfach, 1978), Lecture Notes in Math. 706, Springer, Berlin 1979, 41–53. 239, 240

[17] T. Byczkowski, Zero-one laws for Gaussian measures on metric abelian groups. *Studia Math.* **69** (1980), 159–189. 239, 240

[18] H. Carnal, Non-validité du théorème de Lévy-Cramer sur le cercle. *Publ. Inst. Statist. Univ. Paris* **13** (1964), 55–56. 238

[19] L. Corwin, Generalized Gaussian measure and a "functional equation": I. *J. Funct. Anal.* **5** (1970), 412–427. 244

[20] L. Corwin, Generalized Gaussian measure and a "functional equation": II. *J. Funct. Anal.* **6** (1970), 481–505. 244

[21] H. Cramer, Über eine Eigenschaft der normalen Verteilungsfunktion. *Math. Z.* **41** (1936), 405–414. 237

[22] R. Cuppens, R. Cuppens, *Decomposition of multivariate probabilities*. Probab. Math. Statist. 29, Academic Press, New York 1975. 237

[23] G. Darmois, Analyse générale des liaisons stochastiques. Etude particulière de l'analyse factorielle linéaire. *Rev. Inst. Internat. Statistique* **21** (1953), 2–8. v, 241

[24] D. Ž. Djoković, A representation theorem for $(X_1 - 1)(X_2 - 1)\dots(X_n - 1)$ and its applications. *Ann. Polon. Math.* **22** (1969), 189–198. 238

[25] G. M. Feldman, On the decomposition of a Gaussian distribution on groups. *Teor. Veroyatnost. i Primenen.* **22** (1977), 136–143; English transl. *Probab. Appl.* **22** (1977), 133–140. 238

[26] G. M. Feldman, Generalized Poisson distribution on groups. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **72** (1977), 161–185; English transl. *J. Sov. Math.* **23** (1983), 2335–2351. 238

[27] G. M. Feldman, Gaussian distributions on locally compact abelian groups. *Teor. Veroyatnost. i Primenen.* **23** (1978), 548–563; English transl. *Theory Probab. Appl.* **23** (1978), 529–542. 237, 238

[28] G. M. Feldman, On Urbanik's characterization of Gaussian measures on locally compact abelian groups. *Studia Math.* **73** (1982), 81–86. 238

[29] G. M. Feldman, Gaussian distributions in the sense of Bernstein on groups. *Teor. Veroyatnost. i Primenen.* **31** (1986), 47–58; English transl. *Theory Probab. Appl.* **31** (1987), 40–49. vi, 239

[30] G.M. Feldman, Marcinkiewicz and Lukacs theorems on abelian groups. *Teor. Veroyatnost. i Primenen.* **34** (1989), 330–339; English transl. *Theory Probab. Appl.* **34** (1989), 290–297. 238

[31] G. M. Feldman, Characterization of the Gaussian distribution on groups by the independence of linear statistics. *Sibirsk. Mat. Zh.* **31** (1990), 180–190; English transl. *Siberian Math. J.* **31** (1990), 336–345. vi, 241

[32] G. M. Feldman, Groups admitting a characterization of the Gaussian distribution by the equidistribution of a monomial and linear statistics. *Ukrain. Mat. Zh.* **42** (1990), 139–142; English transl. *Ukr. Math. J.* **42** (1990), 125–128. 239

[33] G. M. Feldman, *Arithmetic of probability distributions, and characterization problems on Abelian groups*. Transl. Math. Monogr. 116, Amer. Math. Soc., Providence, RI, 1993. 238, 239

[34] G. M. Feldman, On the Skitovich–Darmois theorem on Abelian groups. *Teor. Veroyatnost. i Primenen.* **37** (1992), 695–708; English transl. *Theory Probab. Appl.* **37** (1992), 621–631. 241, 242

[35] G. M. Feldman, On a characterization theorem for the Cauchy distribution on abelian groups. *Sibirsk. Mat. Zh.* **36** (1995), 215–221; English transl. *Siberian Math. J.* **36** (1995), 196–201.

[36] G. M. Feldman, The Skitovich–Darmois theorem for compact groups. *Teor. Veroyatnost. i Primenen.* **41** (1996), 901–906; English transl. *Theory Probab. Appl.* **41** (1997), 768–773. 242

[37] G. M. Feldman, The Skitovich–Darmois theorem for discrete periodic Abelian groups. *Teor. Veroyatnost. i Primenen.* **42** (1997), 747–756; English transl. *Theory Probab. Appl.* **42** (1998), 611–617. 242

[38] G. M. Feldman, More on the Skitovich–Darmois theorem for finite Abelian groups. *Teor. Veroyatnost. i Primenen.* **45** (2000), 603–607; English transl. *Theory Probab. Appl.* **45** (2001), 507–511. 242

[39] G. M. Feldman, On groups for which the Bernstein characterization of a Gaussian distribution is valid (Ukrainian). *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **2001**, no.8, 29–32. 239

[40] G. M. Feldman, A characterization of the Gaussian distribution on abelian groups. *Probab. Theory Related Fields* **126** (2003), 91–102. 241

[41] G. M. Feldman, On the Heyde theorem for finite abelian groups. *J. Theoret. Probab.* **17** (2004), 929–941. 244

[42] G. M. Feldman, On a characterization theorem for locally compact abelian groups. *Probab. Theory Related Fields* **133** (2005), 345–357. 244

[43] G. M. Feldman, On the Heyde theorem for discrete abelian groups. *Studia Math.* **177** (2006), 67–79. 244

[44] G. M. Feldman, Independent linear statistics on a-adic solenoids. *Dokl. Akad. Nauk* **417** (2007), 464–467; English transl. *Doklady Math.* **76** (2007), 904–907. vi, 243

[45] G. M. Feldman and A. E. Fryntov, On the decomposition of the convolution of a Gaussian and Poisson distribution on locally compact abelian groups. *J. Multivariate Anal.* **13** (1983), 148–166. 238

[46] G. M. Feldman and P. Graczyk, On the Skitovich–Darmois theorem on compact abelian groups. *J. Theoret. Probab.* **13** (2000), 859–869. vi, 242

[47] G. M. Feldman and P. Graczyk, On the Skitovich–Darmois theorem for discrete abelian groups. *Teor. Veroyatn. Primen.* **49** (2004), 596–601; English transl. *Theory Probab. Appl.* **49** (2005), 527–531. vi, 242

[48] W. Feller, *An introduction to probability theory and its applications*. Vol. I, 3rd. ed., John Wiley & Sons, New York 1968. 39

[49] U. Franz, D. Neuenschwander, and R. Schott, Gauss laws in the sense of Bernstein and uniqueness of embedding into convolution semigroups on quantum groups and braided groups. *Probab. Theory Related Fields* **109** (1997), 101–127. vi

[50] L. Fuchs, *Infinite abelian groups*. Vol. I, Pure Applied Math. 36-I, Academic Press, New York 1970. 1, 10, 166, 167, 168

[51] L. Fuchs, *Infinite abelian groups*. Vol. II, Pure Applied Math. 36-II, Academic Press, New York 1973. 1, 3, 4, 11, 149, 167, 233

[52] S. S. Gabrielyan, On a characterization of the Cauchy distribution on groups by the equidistribution of a monomial and a linear statistic. *Mat. Fiz. Anal. Geom.* **1** (1994), 410–415 (in Russian). vi

[53] A. O. Gel'fond, *Calculus of finite differences*. Hindustan Publishing, Delhi 1971. 70

[54] S. G. Ghurye and I. Olkin, A characterization of the multivariate normal distribution. *Ann. Math. Statist.* **33** (1962), 533–541. v, 139, 241

[55] P. Graczyk and G. M. Feldman, Independent linear statistics on finite abelian groups. *Ukraïn. Mat. Zh.* **53** (2001), 441–448; English transl. *Ukrainian Math. J.* **53** (2001), 499–506. vi, 242, 243

[56] P. Graczyk and J.-J. Loeb, A Bernstein property of measures on groups and symmetric spaces. *Probab. Math. Statist.* **20** (2000), 141–149. vi, 240

[57] U. Grenander, *Probability on algebraic structures*. John Wiley & Sons, New York 1963. 11, 13, 238

[58] J. Hájek, On a property of normal distribution of any stochastic process. *Czechoslovak Math. J.* **8** (83) (1958), 610–618; English transl. *Select. Transl. Math. Statist. and Probability*, Vol. 1, Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I., 245–252. 238

[59] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I, Grundlehren Math. Wiss. 115, Spinger-Verlag, Berlin 1963. 1, 3, 4, 5, 6, 7, 8, 9, 10, 16, 28, 216, 219, 223, 226, 227, 234

[60] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. II, Grundlehren Math. Wiss. 152, Spinger-Verlag, Berlin 1970. 14, 15

[61] C. C. Heyde, Characterization of the normal law by the symmetry of a certain conditional distribution. *Sankhyā Ser. A* **32** (1970), 115–118. v, 243

[62] H. Heyer, Factorization of probability measures on locally compact groups, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **8** (1967), 231–258. 238

[63] H. Heyer, *Probability measures on locally compact groups*, Ergeb. Math. Grenzgeb. (2) 94, Springer-Verlag, Berlin 1977. 237

[64] H. Heyer and Ch. Rall, Gaußsche Wahrscheinlichkeitsmaße auf Corwinschen Gruppen. *Math. Z.* **128** (1972), 343–361. vi, 239

[65] M. Kac, On a characterization of the normal distribution. *Amer. J. Math.* **61** (1939), 726–728. v, 239

[66] A. M. Kagan, Yu. V. Linnik, and C. R. Rao, *Characterization problems in mathematical statistics*. Wiley Ser. Probab. Math. Statist., John Wiley & Sons, New York 1973. 239

[67] S. Kakutani, On equivalence of infinite product measures. *Ann. of Math.* (2) **49** (1948), 214–224. 28

[68] S. F. Kožuhov, Regularly complete abelian groups. *Izv. Vyssh. Uchebn. Zaved. Mat.* **1980**, no. 12, 14–19; English transl. *Sov. Math. (Iz. VUZ)* **24** (1980), no. 4, 14–22. 223

[69] K. Kuratowski, *Topology*. Vol. I, new edition, revised and augmented, Academic Press, New York; PWN – Polish Scientific Publishers, Warszawa 1966. 12

[70] K. Lajkó, On the functional equation $f(x)g(y) = h(ax+by)k(cx+dy)$. *Period. Math. Hungar.* **11** (1980), 187–195. 241

[71] M. Laczkovich, Polynomial mappings on abelian groups. *Aequationes Math.* **68** (2004), 177–199. 238

[72] Yu. V. Linnik, A remark on Cramer's theorem on the decomposition of the normal law. *Teor. Veroyatnost. i Primenen.* **1** (1956), 479–480; English transl. *Theory Probab. Appl.* **1** (1956), 435–436. 241

[73] Yu. V. Linnik, *Decomposition of probability distributions*. Oliver & Boyd, Edinburgh 1964. 237

[74] Ju. V. Linnik and I. V. Ostrovskiĭ, *Decomposition of random variables and vectors*. Transl. Math. Monographs 48, Amer. Math Soc., Providence, RI, 1977. 18, 47, 191, 237

[75] L. Z. Līvšic, I. V. Ostrovskiĭ, and G. P. Čistjakov, The arithmetic of probability laws. In *Probability theory, Mathematical statistics, Theoretical cybernetics*, vol. 12, Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow 1975, 5–42 (in Russian). 237

[76] E. Lukacs, *Characteristic functions*. Second edition, revised and enlarged, Hafner Publishing Co., New York 1970. 237

[77] J. Marcinkiewicz, Sur les variables aléatoires enroulées, *C. R. Séance et Conf. Soc. Math. France* **1938**, C. R. Soc. Math. France 1938, 34–36. 238

[78] M. V. Mironyuk, On the Bernstein functional equation. *Mat. Fiz. Anal. Geom.* **11** (2004), 177–188 (in Russian). vi, 239

[79] M. V. Myronyuk, On the Skitovich–Darmois theorem on abelian groups. *Ukraïn. Mat. Zh.* **56** (2004), 1342–1356; English transl. *Ukrainian Math. J.* **56** (2004), 1602–1618. 242, 243

[80] M. Myronyuk, An analogue of the Bernstein theorem for the cylinder. *Aequationes Math.* **71** (2006), 54–69. 239

[81] M. V. Myronyuk and G. M. Feldman, On a characterization theorem on finite abelian groups. *Sibirsk. Mat. Zh.* **46** (2005), 403–415; English transl. *Siberian Math. J.* **46** (2005), 315–324. 244

[82] M. V. Myronyuk and G. M. Feldman, On the Skitovich–Darmois theorem on the two-dimensional torus. *Dokl. Akad. Nauk* **401** (2005), 741–743; English transl. *Doklady Math.* **71** (2005), 274-276. vi, 241

[83] M. V. Mironyuk and G. M. Feldman, The Skitovich–Darmois equation in the classes of continuous and measurable functions. *Dokl. Akad. Nauk* **410** (2006), 453–456; English transl. *Doklady Math.* **74** (2006), 700–703. 244

[84] M. V. Myronyuk and G. M. Feldman, Independent linear statistics on the two-dimensional torus. *Teor. Veroyatn. Primen.* **52** (2007), 3–20; English transl. *Theory Probab. Appl.* **52** (2008), 78–92. vi, 241

[85] D. Neuenschwander, Gauss measures in the sense of Bernstein on the Heisenberg group. *Probab. Math. Statist.* **14** (1993), 253–256. 240

[86] D. Neuenschwander, B. Roynette, and R. Schott, Characterizations of gaussian distributions on simply connected nilpotent Lie groups and symmetric spaces. *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), 87–92. 240

[87] D. Neuenschwander and R. Schott, The Bernstein and Skitovič–Darmois characterization theorems for Gaussian distributions on groups, symmetric spaces, and quantum groups. *Exposition. Math.* **15** (1997), 289–314. vi

[88] I. V. Ostrovskiĭ, The arithmetic of probability distributions. *J. Multivariate Anal.* **7** (1977), 475–490. 237

[89] K. R. Parthasarathy, *Probability measures on metric spaces*. Probab. Math. Statist. 3, Academic Press, New York 1967. 11, 16, 17, 19, 237

[90] K. R. Parthasarathy, R. Ranga Rao, and S. R. S. Varadhan, Probability distributions on locally compact abelian groups. *Illinois J. Math.* **7** (1963), 337–369. 237

[91] B. L. S. Prakasa Rao, A characterization of Gaussian measures on locally compact abelian groups. *Mat. Zametki* **22** (1977), 759–762; English transl. *Math. Notes* **22** (1978), 914–916. vi

[92] B. Ramachandran, *Advanced theory of characteristic functions*. Statistical Publishing Society, Calcutta 1967. 237

[93] A. L. Rukhin, On a theorem of S. N. Bernshtein. *Mat. Zametki* **6** (1969), 301–307; English transl. *Math. Notes* **6** (1969), 638–641. vi, 239

[94] A. L. Ruhin, Some statistical and probability problems on groups. *Trudy Mat. Inst. Steklov.* **111** (1970), 52–109. English transl. *Proc. Steklov Inst. Math.* **111** (1970), 59–129. vi, 239

[95] V. V. Sazonov and V. N. Tutubalin, Probability distributions on topological groups. *Teor. Veroyatnost. i Primenen.* **11** (1966), 3–55; English transl. *Theor. Probab. Appl.* **11** (1966), 1–45. 237, 238

[96] K. Schmidt, On a characterisation of certain infinitely divisible positive definite functions and measures. *J. London Math. Soc.* (2) **4** (1972), 401–407. 241

[97] E. Siebert, Einige Bemerkungen zu den Gauß-Verteilungen auf lokalkompakten abelschen Gruppen. *Manuscripta Math.* **14** (1974), 41–55. 237

[98] V. P. Skitovič, On a property of the normal distribution. *Dokl. Akad. Nauk SSSR* (*N.S.*) **89** (1953), 217–219 (in Russian). v, 241

[99] L. Székelyhidi, *Convolution type functional equations on topological abelian groups*. World Scientific Publishing, Singapore 1991. 226

[100] K. Urbanik, Gaussian measures on locally compact abelian topological groups. *Studia Math.* **19** (1960), 77–88. 49, 238

Symbol index

Subject index